Advanced Microeconomics II

Game Theory

Jiaming Mao

School of Economics, XMU

Introduction

Game theory is the study of *strategic* interaction among *multiple*, *rational* agents.

- The outcomes affecting a person depend not only on the person's own action, but on the actions of others.
- Individuals choose their best actions while taking into account the actions that others might take.
  - need to understand what others will do
  - need to understand what others think you will do
  - . . .

Introduction

A **game** is a multi-person decision problem.

- Games can be *cooperative* or *non-cooperative*.
- Non-cooperative games can be *static* (strategic) or *dynamic* (extensive).
- Games can have *complete* information or *incomplete* information.

This course will focus on non-cooperative games.

Prize Game

A game with 3 prize levels (1st, 2nd, 3rd) is played as follows:

- On a piece of paper, write down either “A” or “B”.
- You will then be paired randomly with another student.

Prizes are distributed according to:

		Your Pair	
		A	B
You	A	3rd, 3rd	1st, no prize
	B	no prize, 1st	2nd, 2nd

Prize Game: Version 1

		Your Pair	
		A	B
You	A	3rd, 3rd	1st, no prize
	B	no prize, 1st	2nd, 2nd

Outcome Matrix

⟶

		Your Pair	
		A	B
You	A	1,1	4,0
	B	0,4	3,3

Payoff Matrix

- Both you and your pair are "selfish" (care only about own winnings).
- Writing down "A" is a *strictly dominant* strategy.
- Rational choice can lead to sub-optimal outcomes.

Prize Game: Version 2

		Your Pair	
		A	B
You	A	3^{rd}, 3^{rd}	1^{st}, no prize
	B	no prize, 1^{st}	2^{nd}, 2^{nd}

Outcome Matrix

⟶

		Your Pair	
		A	B
You	A	1,1	0,0
	B	0,0	3,3

Payoff Matrix

- Both you and your pair are “altruistic” (feel bad when the other person doesn’t win a prize).
- No strictly dominant strategies.

Prize Game: Version 3

		Your Pair	
		A	B
You	A	3^{rd}, 3^{rd}	1^{st}, no prize
	B	no prize, 1^{st}	2^{nd}, 2^{nd}

Outcome Matrix

$\longrightarrow$

		Your Pair	
		A	B
You	A	1,1	0,0
	B	0,4	3,3

Payoff Matrix

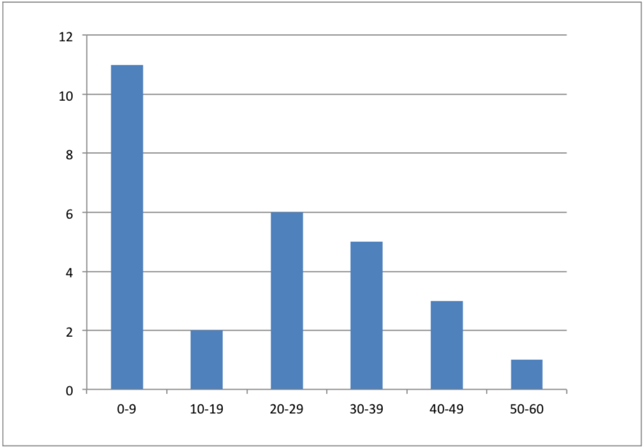
- You are “altruistic” and your pair is “selfish.”
- “A” is a strictly dominant strategy for your pair.

Pick a Number

Write down an integer number between 1 and 100.The'winner in this game is the person whose number is closest to $\frac{2}{3}$ of the class average.

Pick a Number

- Class average: 20
- Winning bid: 13
- Distribution:



Strategic (Normal) Form Game

Definition

A strategic form game is $G = (S_i, u_i)_{i=1}^{N}$, where

- $\{1, \ldots, N\}$ is the set of players,
- S_i is the set of strategies available to player i,
- $u_i : S_1 \times \cdots \times S_N \to \Re$ is player i's payoff function.

G is finite if each S_i is finite.

- Let $S \equiv S_1 \times \cdots \times S_N$ be the strategy space. Denote $s_i \in S_i$ as a strategy chosen by player i, and $s \equiv (s_1, \ldots, s_N) \in S$ as a *strategy profile.*
- Each player's payoff is a function of the strategies chosen by all players.

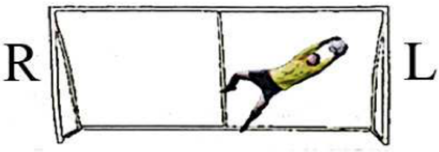
Matching Pennies

Each player has a penny. They each secretly choose a side of the coin to reveal and then they reveal their coins simultaneously.

- If the faces match, the second player gives the first player \$1. If the faces do not match, the first player gives the second player \$1.
- Payoffs:

		Player 2	
		H	T
Player 1	H	1,-1	-1,1
	T	-1,1	1,-1

Penalty Kick (v1)



The penalty taker needs to decide whether to kick to the left or to the right. Meanwhile, the goalie needs to decide whether to dive to the left or to the right[1].

- If the goalie guesses the right direction, he/she saves the penalty. Othewise the kicker scores.
- Payoffs:

		Goalie	
		L	R
Kicker	L	0,0	1,-1
	R	1,-1	0,0

---

[1] Left or right defined from the penalty taker's perspective

Battle of the Sexes

Tom and Lucy would like to meet for dinner, but must decide whether to go to Lucy's favorite restaurant for rice, or Tom's favorite restaurant for noodles.

- Payoffs:

		Tom	
		Rice	Noodles
Lucy	Rice	4,1	0,0
	Noodles	0,0	1,4

Chicken

Two car drivers play “chicken” - they start driving head-on towards each other and choose whether or not to swerve.

- Payoffs:

		Player 2	
		Keep going	Swerve
Player 1	Keep going	-100,-100	10,-10
	Swerve	-10,10	0,0

Hawk-Dove

Two animals are contesting for a resource with value V. Each animal can act in either an aggressive (Hawk), or peaceful (Dove) manner.

- If both are hawks, they will fight. Each wins with probabiliy $\frac{1}{2}$ and pays a cost C.
- If both are doves, they share the resource evenly.
- If one is a hawk and the other a dove, the hawk takes the entire resource.

Prisoner's Dilemma

Two robbery suspects are arrested and questioned by police in separate rooms. The police only have the evidence to charge them for trespassing, but not have enough evidence to charge them for robbery. A deal is offered to the prisoners: each is given the opportunity either to testify that the other committed the robbery, or to remain silent.

- If A testifies against B and B remains silent, then A is set free and B gets 12 months in jail.
- If both remain silent, each will get 1 month in jail.
- If both testify, then they will each get 10 months in jail.
- Payoffs:

		Prisoner 2	
		silent	testify
Prisoner 1	silent	-1,-1	-12,0
	testify	0,-12	-10,-10

Strictly Dominant Strategies

Let $s_{-i} \equiv (s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_N)$ be the strategy choices of all players except player i.

Definition (Strictly Dominant Strategies)

A strategy, $\widetilde{s}_i$, for player i is strictly dominant if $u_i\left(\widetilde{s}_i, s_{-i}\right) > u_i\left(s_i, s_{-i}\right)$ for all $(s_i, s_{-i}) \in S$ with $s_i \neq \widetilde{s}_i$

- Always play the strictly dominant strategy if it exists!

Newcomb's Problem

In front of you are two boxes, A and B. You call see that in box B there is £1000, but you cannot see what is in box A. You have two choices: (1) take just box A; (2) take both A and B.

At the same time, a demon has predicted whether you will take just one box or take two boxes. The demon is very good at predicting these things – in the past she has made many similar predictions and been right every time. If the demon predicts that you will take both boxes, then she's put nothing in box A. If the demon predicts you will take just one box, she has put £1,000,000 in box A.

Newcomb's Problem

		Demon	
		Predicts 1 box	Predicts 2 boxes
You	Take 1 box	£1,000,000	£0
	Take 2 boxes	£1,000,000	£1,000

(Your) Outcome Matrix

		Demon	
		Predicts 1 box	Predicts 2 boxes
You	Take 1 box	1000,1	0,0
	Take 2 boxes	1001,0	1,1

Payoff Matrix

Strictly Dominated Strategies

Note (Strictly Dominated Strategy)

Let $s_i \in S_i$. If $\exists s_i' \in S_i$ such that $u_i\left(s_i', s_{-i}\right) > u_i\left(s_i, s_{-i}\right)\ \forall s_{-i} \in S_{-i}$, then s_i is a strictly dominated strategy.

- We will provide a formal definition later when we introduce mixed strategies.

Iterative Deletion of Strictly Dominated Strategies (IDSDS)

	L	M	R
U	3,0	0,-5	0,-4
C	1,-1	3,3	-2,4
D	2,4	4,1	-1,8

⟶

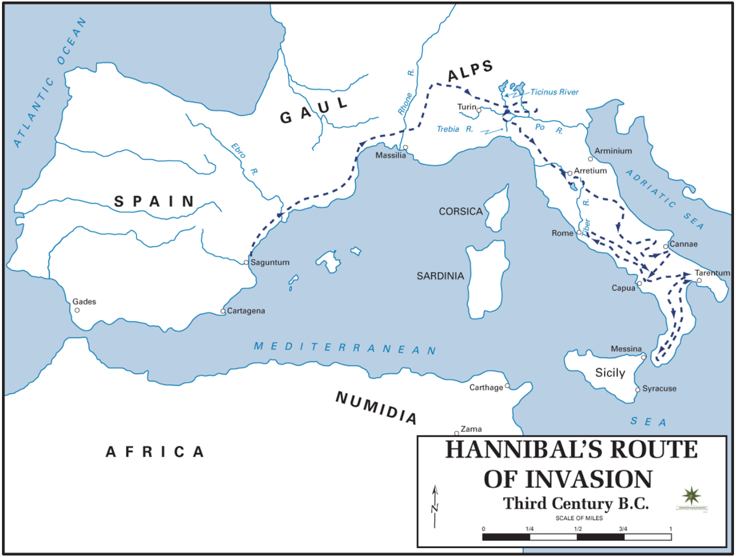
	L	R
U	3,0	0,-4
D	2,4	-1,8

↓

	L
U	3,0

Hannibal

Hannibal


ATLANTIC OCEAN
GAUL
ALPS
Ticinus River
Turin
Trebia R.
Po R.
Rhone R.
Ebro R.
Massilia
SPAIN
Saguntum
Gades
Cartagena
CORSICA
SARDINIA
Arminium
Arretium
Tiber R.
Rome
Cannae
Tarentum
Capua
ADRIATIC SEA
Messina
Sicily
Syracuse
MEDITERRANEAN SEA
Carthage
NUMIDIA
Zama
AFRICA
HANNIBAL'S ROUTE OF INVASION
Third Century B.C.
N
SCALE OF MILES
0
1/4
1/2
3/4
1

Hannibal

- Hannibal is planning to attack Rome with an army of 2 battalions. There are 2 routes through which he can lead his army. One is a hard route (from Iberia to Northern Italy through the Alps) and the other is an easy route (from Carthage to Southern Italy by sea). If he chooses the hard route, he will lose 1 battalion on the way. If he chooses the easy route, he will arrive in Rome with his army intact.
- You are a Roman general in charge of defending Rome. You can only defend one of these routes and must decide which one to defend.
- Payoffs:

		Hannibal	
		E	H
You	E	1,1	1,1
	H	0,2	2,0

Weakly Dominated Strategies

Note (Weakly Dominated Strategy)

Let $s_i \in S_i$. If $\exists s_i' \in S_i$ such that $u_i\left(s_i', s_{-i}\right) \geq u_i\left(s_i, s_{-i}\right)\ \forall s_{-i} \in S_{-i}$, with at least one strict inequality, then s_i is a weakly dominated strategy.

- We will provide a formal definition later when we introduce mixed strategies.

Iterative Deletion of Weakly Dominated Strategies (IDWDS)

	L	R
A	3,4	4,3
B	5,3	3,5
C	5,3	4,3

- The order of deletion can matter when iteratively deleting weakly dominated strategies.

Common Knowledge

- If players are rational, they won't pick $68 - 100$.
- If players know the other players are rational, they won't pick $46 - 67$
- If players know the other players know that the other players are rational, they won't pick $31 - 45$, ...

Definition (Common Knowledge)

In a game, p is common knowledge if each player knows it, each player knows that each player knows it, each player knows that each player knows that each player knows it, and so on.

- Iterative deletion of dominated strategies requires not just rationality, but common knowledge in rationality.

Common Knowledge

		Player 2		
		L	M	R
Player 1	U	2,2	1,1	4,0
	D	1,2	4,1	3,5

- Common Knowledge solution: (U,L)
- Alternative model of knowledge:
  - Both player 1 and 2 are rational.
  - Player 1 thinks that player 2 is clueless and randomizes across his strategies with equal probability.
  - Player 2 thinks that player 1 is rational and that player 1 thinks he is randomizing.
- Solution: (D,R)

Coordinated Attack

- Two divisions of an army are waiting for decisions to attack the enemy. If both divisions attack simultaneously they will win the battle, whereas if only one division attacks it will be defeated. Neither general will attack unless he is sure that other will attack with him.
- Commander A is in peace negotiations with the enemy. The generals agreed that if the negotiations fail, commander A will send a message to commander B with an attack plan.
- However, there is a small probability ε that the messenger gets intercepted and the message does not arrive. The messenger takes one hour normally. How long will it take to coordinate on the attack?

Mutual Knowledge v. Common Knowledge

In Slapville, it is culturally required to slap oneself if one is in public with a dirty face. Larry, Curly and Moe are in a room together, fortunately one without mirrors. Each of them has a dirty face, but they can't see their own faces, they can only see the other faces. And each face is dirty. Inspector Renault walks into the room and says, "I'm shocked! Someone in this room has a dirty face." After a long delay, Larry, Curly and Moe each slap themselves in the face. Why?

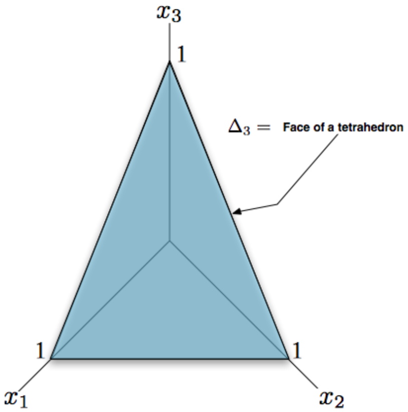
Mixed Strategy

Definition (Mixed Strategy)

Given $G = (S_i, u_i)_{i=1}^N$, a mixed strategy σ_i for player i is a probability distribution over S_i, $\sigma_i : S_i \to [0,1]$, that assigns to each $s_i \in S_i$ the probability $\sigma_i\left(s_i\right)$ that s_i will be played. The set of mixed strategies for player i is $\Delta_i \equiv \left\{\sigma_i : S_i \to [0,1] \,\middle|\, \sum_{s_i \in S_i} \sigma_i\left(s_i\right) = 1 \right\}$. The mixed strategy space of G is $\Delta \equiv \Delta_1 \times \cdots \times \Delta_N$.

Mixed Strategy

Let $S_i = \{s_i^1, \ldots, s_i^J\}$. Then the mixed strategy set Δ_i is the face of the $J-$dimenstional simplex whose basis are the pure strategies $\{s_i^1, \ldots, s_i^J\}$.



Mixed Strategy

Definition (Mixed Strategy Payoff Function)

If u_i is a von Neumann-Morgenstern utility function on S, then

$$u_i\left(\sigma\right) \equiv \sum_{s \in S} \sigma_1\left(s_1\right) \cdots \sigma_N\left(s_N\right) u_i\left(s\right)$$

Matching Pennis

		Player 2	
		H	T
Player 1	H	1,-1	-1,1
	T	-1,1	1,-1

- The mixed strategy σ_i of playing H with 50% probability and T with 50% probabiliy can be specified as $\sigma_i(H) = \frac{1}{2}, \sigma_i(T) = \frac{1}{2}$. Equivalently, we can write $\sigma_i = \frac{1}{2}H + \frac{1}{2}T$ or $\sigma_i = \left(\frac{1}{2}, \frac{1}{2}\right)$.
- The pure strategies of playing H and T are $\sigma_i = (1, 0)$ and $(0, 1)$.

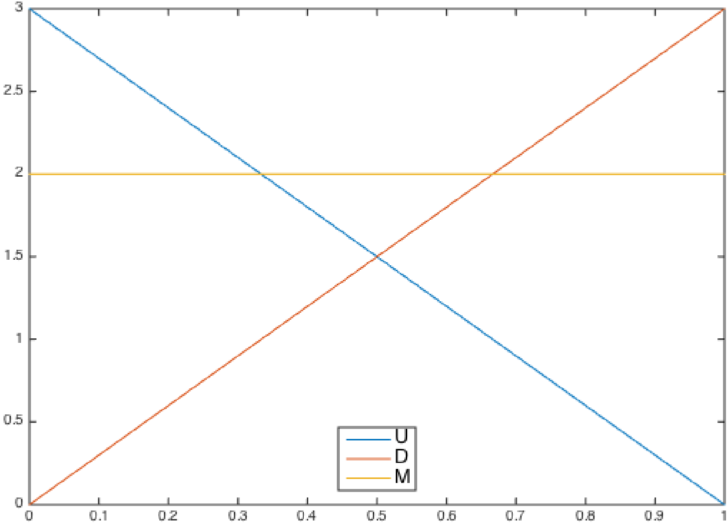
Mixed Strategy: Interpretations

- Actual randomizing behavior
- Other players' *common* belief about the probability that a player plays each pure strategy
- Proportions of the population that play each pure strategy

Best Response

	L	R
U	3,0	0,0
M	2,0	2,0
D	0,0	3,0

Best Response



Best Response

Definition (Best Response)

A strategy $\sigma_i \in \Delta_i$ is a best response to the strategy profile $\sigma_{-i} \in \Delta_{-i}$ if

$$u_i\left(\sigma_i, \sigma_{-i}\right) \geq u_i\left(\sigma_i', \sigma_{-i}\right) \;\; \forall \sigma_i \in \Delta_i$$

A strategy $\sigma_i \in \Delta_i$ is never a best response if $\nexists \sigma_{-i} \in \Delta_{-i}$ for which σ_i is a best response.

Definition (Best Response Function)

A best response function BR_i for player i is a mapping $\Delta_{-i} \to \Delta_i$ defined by

$$BR_i\left(\sigma_{-i}\right) = \left\{\sigma_i \in \Delta_i : u_i\left(\sigma_i, \sigma_{-i}\right) \geq u_i\left(\sigma_i', \sigma_{-i}\right) \;\; \forall \sigma_i' \in \Delta_i\right\}$$

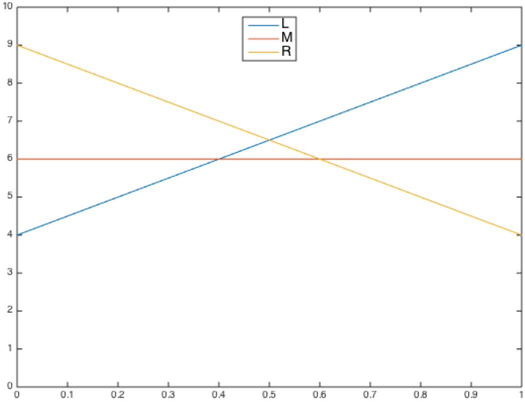
Penalty Kick (v2)

The penalty taker decides whether to kick to the left, to the middle, or to the right. Meanwhile, the goalie decides whether to dive to the left or to the right.

- If the goalie guesses the correct direction, she saves the penalty with 40% probability.
- If the penalty taker kicks to the left or right and the goalie dives to the other direction, the penalty taker scores with probability 90%
- If the penalty taker kicks to the middle, she scores with probability 60%.

	Goalie	
	L	R
L	4,-4	9,-9
M	6,-6	6,-6
R	9,-9	4,-4

Penalty Kick (v2)



- Do not shoot to the middle (i.e. do not play a strategy that is never a BR).

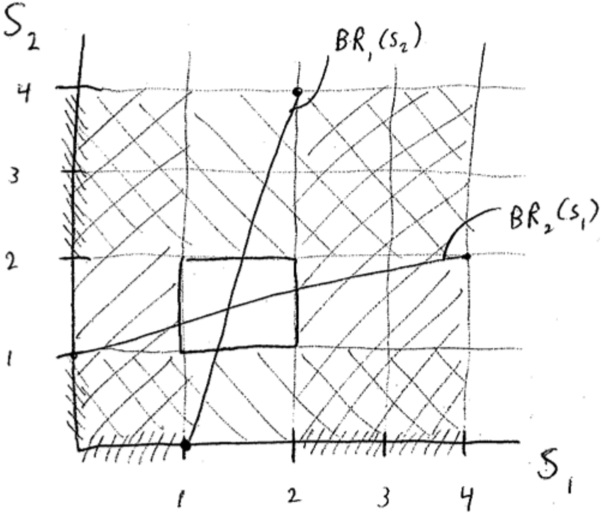
Team Work

- 2 individuals work on a project together. Each chooses effort level $s_i \in [0, 4]$.
- Return to project: $y = 4\left(s_1 + s_2 + bs_1 s_2\right)$
- Payoffs: $u_i\left(s_1, s_2\right) = \frac{1}{2}y - s_i^2$

$\Rightarrow$

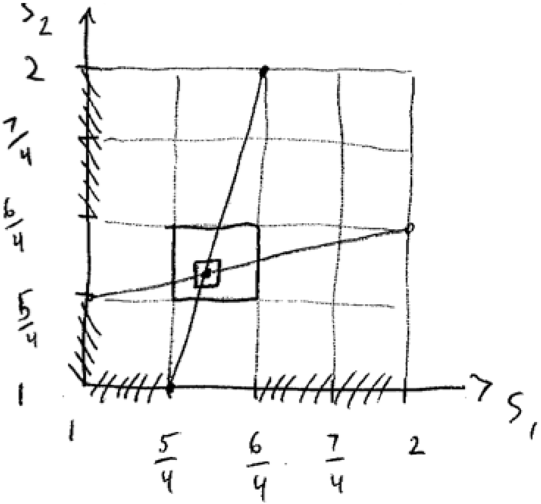
$$\begin{aligned} BR_1\left(s_2\right) &= 1 + bs_2 \\ BR\left(s_1\right) &= 1 + bs_1 \end{aligned}$$

Team Work



Assume $b = \frac{1}{4}$

Team Work



Strictly Dominated Strategies

Definition (Strictly Dominated Strategies)

A strategy σ_i is strictly dominated if $\exists \sigma_i'$ such that

$$u_i\left(\sigma_i', \sigma_{-i}\right) > u_i\left(\sigma_i, \sigma_{-i}\right) \ \forall \sigma_{-i} \in \Delta_{-i}$$

Iterately Strictly Undominated Strategies (ISUS)

Definition (Iteratively Strictly Undominated Strategies)

The set of strategy profiles surviving IDSDS[a] is
$\Delta^{(\infty)} = \Delta_1^{(\infty)} \times \ldots \times \Delta_N^{(\infty)}$, where $\Delta_i^{(\infty)} = \cap_{k=1}^{\infty} \Delta_i^{(k)}$, and

$$\begin{aligned}
\Delta_i^{(0)} &= \Delta_i \\
\Delta_i^{(k)} &= \left\{ \sigma_i \in \Delta_i^{(k-1)} \middle| \nexists \sigma_i' \in \Delta_i^{(k-1)} \right. \\
&\qquad \left. , \text{such that } u_i\left(\sigma_i', \sigma_{-i}\right) > u_i\left(\sigma_i, \sigma_{-i}\right) \ \forall \sigma_{-i} \in \Delta_{-i}^{(k-1)} \right\}
\end{aligned}$$

---

[a] Iterative Deletion of Strictly Dominated Strategies

Rationalizability

Definition (Rationalizability)

The set of strategy profiles surviving IDNBR[a] is $\Psi^{(\infty)} = \Psi_1^{(\infty)} \times \ldots \times \Psi_N^{(\infty)}$, where $\Psi_i^{(\infty)} = \cap_{k=1}^{\infty} \Psi_i^{(k)}$, and

$$\begin{aligned} \Psi_i^{(0)} &= \Delta_i \\ \Psi_i^{(k)} &= \left\{ \sigma_i \in \Psi_i^{(k-1)} \middle| \exists \sigma_{-i} \in \Psi_{-i}^{(k-1)} \right. \\ & \left. \quad , \text{such that } u_i\left(\sigma_i, \sigma_{-i}\right) \geq u_i\left(\sigma_i', \sigma_{-i}\right) \ \forall \sigma_i' \in \Psi_i^{(k-1)} \right\} \end{aligned}$$

---

[a] Iterative Deletion of Never Best Responses

Best Response and Strict Dominance

Proposition

If a strategy is strictly dominated[a]*, then it is never a best response .*

---

[a] by any pure or mixed strategies

- In a 2-player game, if a strategy is never a best response, it is strictly dominated.
- The set of strictly dominated strategies is a subset of never best response strategies. The two sets are equal in 2-player games.

Rationalizability and IDSDS

Proposition

If a strategy does not survive IDSDS, then it is not rationalizable .

- The set of rationalizable strategies is a subset of ISUS.
    - In 2-player games, the two sets are equal.
- Rationalizability is (weakly) more restrictive than IDSDS.

Pure Strategy Nash Equilibrium (PSNE)

Definition (Pure Strategy Nash Equilibrium)

Given $G = (S_i, u_i)_{i=1}^{N}$, the strategy profile $s^* \in S$ is a pure strategy Nash equilibrium if for each player i,

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right)\ \forall s_i \in S_i$$

Nash Equilibrium (NE)

Definition (Nash Equilibrium)

A strategy profile $\sigma^* \in \Delta$ is a Nash equilibrium if for each player i,

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(\sigma_i, \sigma_{-i}^*\right) \ \forall \sigma_i \in \Delta_i$$

- Given the strategies chosen by other players, no player can improve her expected payoff by deviating (unilaterally randomising differently).

Nash Equilibrium

Proposition

σ^* is a NE iff $\sigma_i^* \in BR_i\left(\sigma_{-i}^*\right)$ for each player i.

Proposition

if σ^* is a NE, then $\sigma^* \in \Psi^{(\infty)}$.

Proposition

if σ^* is a NE, then $\sigma^* \in \Delta^{(\infty)}$.

Weakly Dominated Strategies

Definition (Weakly Dominated Strategies)

A strategy σ_i is weakly dominated if $\exists \sigma_i'$ such that

$$u_i\left(\sigma_i', \sigma_{-i}\right) \geq u_i\left(\sigma_i, \sigma_{-i}\right) \ \forall \sigma_{-i} \in \Delta_{-i}$$

, with at least one strict inequality.

Nash Equilibrium and Weak Dominance

	L	R
A	1,1	100,0
B	0,100	100,100

Nash Equilibrium

- No regrets: no individual can do *strictly* better by deviating, holding others' strategies fixed.
- Self-fulfilling beliefs: if everyone in the game believes that everyone else is going to play their part of a particular Nash Equilibrium, then everyone will, in fact, play their part of that Nash Equilibrium.

Cournot Competition

Problem

Two profit-maximizing firms simultaneously choose quantity of production. Both firms have constant marginal cost of production c. Market demand is

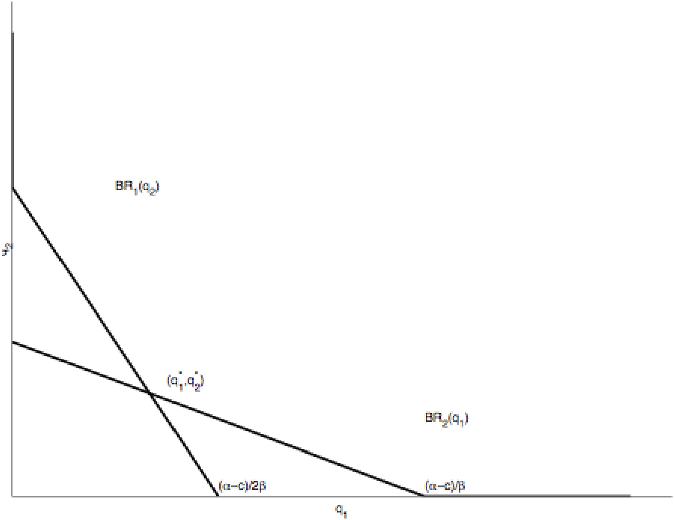
$$P\left(Q\right)=\alpha-\beta Q$$

, where $Q=q_1+q_2$.

$$u_i\left(q_i,q_j\right)=q_i\left(\alpha-\beta q_i-\beta q_j-c\right)$$

$$BR_i\left(q_j\right)=\begin{cases}\frac{\alpha-c}{2\beta}-\frac{q_j}{2} & \text{if } q_j\leq\frac{\alpha-c}{\beta}\\ 0 & \text{o.w.}\end{cases}$$

Cournot Competition



$$q_1^* = q_2^* = \frac{\alpha - c}{3\beta}$$

Bertrand Competition

Problem

Two profit-maximizing firms simultaneously set prices for their products. Both firms have constant marginal cost of production c. Market demand for each firm's product is

$$Q_i\left(p_1, p_2\right) = \begin{cases} a - bp_i & \text{if } p_i < p_j \\ \frac{1}{2}\left(a - bp_i\right) & \text{if } p_i = p_j \\ 0 & \text{if } p_i > p_j \end{cases}$$

- NE: (c, c)

Voting Game

Problem

Three players simultaneously cast ballots for one of three alternatives A, B, or C. If a majority chooses any policy that policy is implemented. If the votes split 1-1-1, we assume that A will be implemented. Suppose the preferences are:

$$u_1(A) > u_1(B) > u_1(C)$$
$$u_2(B) > u_2(C) > u_2(A)$$
$$u_3(C) > u_3(A) > u_3(B)$$

- NE: $(A, A, A), (B, B, B), (C, C, C), (A, B, A), (A, C, C)$.

Wage Bargaining

- A firm/union wage dispute is being settled by arbitration.
- In final offer arbitration, the firm and union simultaneously make offers, w_f and w_u
- The arbitrator then chooses one of the offers according to which one is closest to her ideal settlement point.
- The firm and union both believe the arbitrator's ideal settlement point x is randomly distributed according to $F(x)$

Wage Bargaining

Union's Objective:

$$\max_{w_u} \left\{ w_f F\left(\frac{w_f + w_u}{2}\right) + w_u \left(1 - F\left(\frac{w_f + w_u}{2}\right)\right) \right\}$$

$\Rightarrow$

$$\frac{1}{2}\left(w_u^* - w_f\right) f\left(\frac{w_f + w_u^*}{2}\right) = 1 - F\left(\frac{w_f + w_u^*}{2}\right)$$

Firm's Objective:

$$\max_{w_f} \left\{ -w_f F\left(\frac{w_f + w_u}{2}\right) - w_u \left(1 - F\left(\frac{w_f + w_u}{2}\right)\right) \right\}$$

$\Rightarrow$

$$\frac{1}{2}\left(w_u - w_f^*\right) f\left(\frac{w_f^* + w_u}{2}\right) = F\left(\frac{w_f^* + w_u}{2}\right)$$

Wage Bargaining

Hence

$$F\left(\frac{w_f^* + w_u^*}{2}\right) = \frac{1}{2} \Rightarrow w_u^* - w_f^* = \frac{1}{f\left(\frac{w_f^*+w_u^*}{2}\right)}$$

Let $x \sim N\left(\mu, \sigma^2\right)^2$, then $\frac{w_f^*+w_u^*}{2} = \mu$,

$$w_u^* = \mu + \sqrt{\frac{\pi}{2}}\sigma, w_f^* = \mu - \sqrt{\frac{\pi}{2}}\sigma$$

---

[2] $f\left(x\right) = \left(2\pi\sigma^2\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left(x-\mu\right)^2\right\}$

Hotelling: Location Choice

Consumers are located uniformly on a single street. Two firms selling an identical product decide where to locate on this street. Each consumer has demand for 1 good and will buy from the firm located nearest to her. If the two firms are at the same location, they will slipt the total demand. How should the two firms choose their locations?

Hotelling: Pricing

- Now fix the two firms' location at the left and right ends of the street. Suppose the street's length is d. A consumer located at $x \in [0, d]$ buying from firm i pays the cost $p_i + k\,|x - z_i|$, where $z_i \in \{0, d\}$ is firm i's location and k is the transportation cost.
- How should each firm set its price p_i?

Hotelling: Pricing

A consumer is indifferent between firm 1 and 2 if

$$p_1 + kx = p_2 + k\left(d - x\right)$$

$\Rightarrow$ position x^* of the indifferent consumer:

$$x^* = \frac{1}{2k}\left(p_2 - p_1\right) + \frac{1}{2}d$$

Hotelling: Pricing

Firm Profits:

$$\begin{aligned}\pi_1 &= (p_1 - c)\, x^* = \frac{p_1}{2k}(p_2 - p_1) + \frac{1}{2} d p_1 \\ \pi_2 &= (p_2 - c)(d - x^*) = \frac{p_2}{2k}(p_1 - p_2) + \frac{1}{2} d p_2\end{aligned}$$

$\Rightarrow$

$$BR_i(p_j) = \frac{1}{2}(p_j + kd + c)$$

$\Rightarrow$

$$p_1^* = p_2^* = kd + c$$

Problem of the Commons

There are N fishermen who go out fishing each day. Every day, each fisherman i chooses how much time t_i to spend catching fish. The cost of fishing per hour is c. The amount of fish they catch per hour is

$$f(T) = \max\left\{0, a - bT^2\right\}, \; T = \sum_i t_i$$

Problem of the Commons: Social Optimum

Social planner optimizes

$$\max_{T} \{ Tf(T) - cT \}$$

$\Rightarrow$

$$T^{SO} = \sqrt{\frac{a-c}{3b}}$$

Problem of the Commons: Nash Equilibrium

Each individual fishermen solves

$$\max_{t_i} \{t_i f(T) - ct_i\}$$

$\Rightarrow$

$$t_i^* = \frac{a - c - bT^2}{2bT}$$

$\Rightarrow$

$$T^{NE} = \sqrt{\frac{a-c}{\left(\frac{2}{N}+1\right)b}} > T^{SO}$$

Checking Mixed Strategy Nash Equilibria

Proposition

σ^* is a NE iff for each player i,

$$u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(s_i, \sigma_{-i}^*\right) \; \forall s_i \in S_i$$

Checking Mixed Strategy Nash Equilibria

Let $supp\left(\sigma_i\right) \equiv \left\{s_i \in S_i \left|\sigma_i\left(s_i\right) > 0\right.\right\}$.

Proposition

σ^* is a NE iff for each player i, $\forall s_i^{'}, s_i^{''} \in supp\left(\sigma_i^*\right)$,

$$u_i\left(s_i^{'}, \sigma_{-i}^*\right) = u_i\left(s_i^{''}, \sigma_{-i}^*\right) = u_i\left(\sigma_i^*, \sigma_{-i}^*\right),$$

and $\forall s_i^{'''} \notin supp\left(\sigma_i^*\right)$, $u_i\left(\sigma_i^*, \sigma_{-i}^*\right) \geq u_i\left(s_i^{'''}, \sigma_{-i}^*\right)$.

Checking Mixed Strategy Nash Equilibria

Equivalently,

Proposition

σ^* is a NE iff for each player i,

$$s_i \in BR_i\left(\sigma^*_{-i}\right)\ \forall s_i \in supp\left(\sigma^*_i\right)$$

- If a mixed strategy is a BR, then each of the pure strateies in the mix must themselves be BR. In particular, each must yield the same expected payoff.

Finding Nash Equilibria

	L	R
U	1,1	0,4
D	0,2	2,1

- No PSNE.
- No NE where one player plays a pure strategy and the other plays a mixed strategy with positive probabilities on both pure strategies.
- Assume player 1 plays U with prob $\alpha \in (0,1)$ and player 2 plays L with prob $\beta \in (0,1)$

$$\begin{aligned} u_1\left(U, \sigma_2^*\right) &= u_1\left(D, \sigma_2^*\right) \Rightarrow \beta = 2\left(1-\beta\right) \\ u_2\left(\sigma_1^*, L\right) &= u_2\left(\sigma_1^*, R\right) \Rightarrow \alpha + 2\left(1-\alpha\right) = 4\alpha + \left(1-\alpha\right) \end{aligned}$$

$$\sigma_1^* = \left(\frac{1}{4}, \frac{3}{4}\right), \sigma_2^* = \left(\frac{2}{3}, \frac{1}{3}\right)$$

Finding Nash Equilibria

	L	M	R
U	1,1	0,2	0,4
C	0,2	5,0	1,6
D	0,2	1,1	2,1

- M is strictly dominated by $\frac{1}{2}L + \frac{1}{2}R$.
- After deleting M, C is strictly dominated by $\frac{2}{3}D + \frac{1}{3}U$.
- $\sigma_1^* = \left(\frac{1}{4}, 0, \frac{3}{4}\right), \sigma_2^* = \left(\frac{2}{3}, 0, \frac{1}{3}\right)$.

Battle of the Sexes

		Tom	
		Rice	Noodles
Lucy	Rice	4,1	0,0
	Noodles	0,0	1,4

- PSNE: (R,R), (N,N).
- To check MSNE, Assume Lucy chooses R with prob $p \in (0,1)$ and Tom chooses R with prob $q \in (0,1)$

$$\begin{aligned} u_L\left(R, \sigma_T^*\right) &= u_L\left(N, \sigma_T^*\right) \Rightarrow 4q = 1 - q \\ u_T\left(\sigma_L^*, R\right) &= u_T\left(\sigma_L^*, N\right) \Rightarrow p = 4\left(1 - p\right) \end{aligned}$$

$$\sigma_L^* = \left(\frac{4}{5}, \frac{1}{5}\right), \sigma_T^* = \left(\frac{1}{5}, \frac{4}{5}\right)$$

- Notice that, for both players, the MSNE payoff is worse than either PSNE payoffs.

Battle of the Sexes

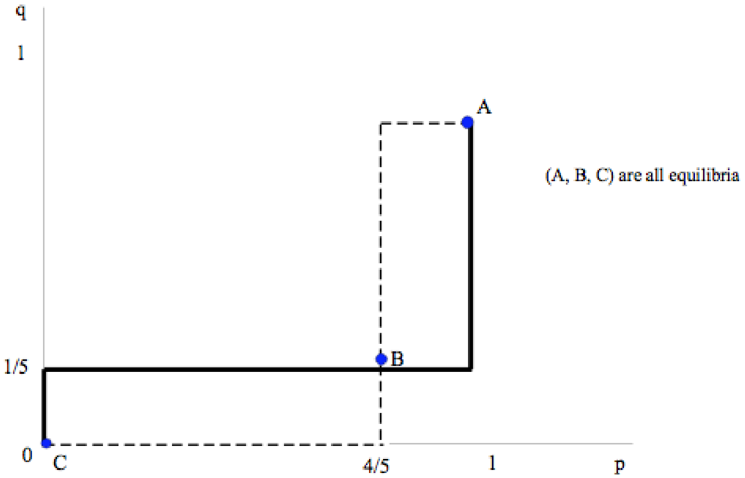
Alteranively, let $p, q \in [0, 1]$,

$$\begin{aligned} u_L(p, q) &= 4pq + (1-p)(1-q) \\ u_T(p, q) &= pq + 4(1-p)(1-q) \end{aligned}$$

$\Rightarrow$

$$BR_L(q) \begin{cases} = 0 & q < \frac{1}{5} \\ \in [0, 1] & q = \frac{1}{5} \\ = 1 & q > \frac{1}{5} \end{cases} \qquad BR_T(p) \begin{cases} = 0 & p < \frac{4}{5} \\ \in [0, 1] & p = \frac{4}{5} \\ = 1 & p > \frac{4}{5} \end{cases}$$

Battle of the Sexes



Tennis

Player A whose turn it is to serve needs to decide whether to serve to the left or to the right of player B. At the same time, Player B needs to decide whether to lean to the right or to the left.

Tennis

		B		
		L	R	
A	L	50,50	80,20	p
	R	90,10	20,80	$1-p$
		q	$1-q$	

- $p^* = 0.7, q^* = 0.6$

Tennis

Now suppose player B's forehand is strengthend:

		B		
		L	R	
A	L	30,70	80,20	p
	R	90,10	20,80	$1-p$
		q	$1-q$	

- **Direct Effect**: B should lean to the left more.
- **Strategic Effect**: A would serve to the left less often, so B should lean to the left less often.
- $p^* = \frac{7}{12}, q^* = \frac{1}{2}$
- So strategic effect > direct effect in this case.

Death in Demascus

Death in Demascus

Consider the story of the man who met Death in Damascus. Death looked surprised, but then recovered his ghastly composure and said, “I am coming for you tomorrow.”

The terrified man that night bought a camel and rode to Aleppo. The next day, Death knocked on the door of the room where he was hiding, and said “I have come for you.”

“But I thought you would be looking for me in Damascus’, said the man. “Not at all,” said Death, “That is why I was supprised to see you yesterday. I knew that today I was to find you in Aleppo.”

Death in Demascus

		Man	
		Damascus	Aleppo
Death	Damascus	1,-1	-1,1
	Aleppo	-1,1	1,-1

- $\sigma_M^* = \left(\frac{1}{2}, \frac{1}{2}\right), \sigma_D^* = \left(\frac{1}{2}, \frac{1}{2}\right).$

Death in Demascus

Now imagine getting to Aleppo is costly:

		Man	
		Damascus	Aleppo
Death	Damascus	1,-1	-1,0.5
	Aleppo	-1,1	1,-1.5

- $\sigma_M^* = \left(\frac{1}{2}, \frac{1}{2}\right), \sigma_D^* = \left(\frac{5}{8}, \frac{3}{8}\right)$.
- In this case strategic effect cancels out direct effect.
  - This will always be the case when only Man's payoffs are changed.
- A player's payoffs determine the *other* player's equilibrium mix.

Paying Taxes

A taxpayer decides whether to honestly pay his taxes or cheat. At the same time, a tax auditor decides whether to audit the tax payer or not.

- Payoffs:

		Taxpayer		
		Honest	Cheat	
Auditor	Audit	2,0	4,-10	p
	Not Audit	4,0	0,4	$1-p$
		q	$1-q$	

- $p^* = \frac{2}{7}, q^* = \frac{2}{3}$
- If we increase the fine of cheating from 10 to 20, will it change the compliance rate of taxpayers?

Investment

Problem

Everyone in a group has a choice between Investing and Not Investing \$1. The return to anyone who doesn't invest is \$0. If 90% or more of the group Invests, then those who Invest get \$2 in return. If less than 90% of the group Invests, then those who invest get \$0 in return.

- If payoff = monetary profit, then NE:
  - Everyone invests.
  - No one invests.

Choosing a Town (v1)

- A population of N people simultaneously choose to live in one of two towns: East Town and West Town. Each town can accomodate all N people.
- All individuals prefer to live in a town with more people. The payoff of living in a town with n people is

$$u_i\left(n\right) = \frac{n}{N}$$

Choosing a Town (v1)

- NE:
  - (stable) All people live in one town.
  - (weak) Each person plays a $\left(\frac{1}{2}, \frac{1}{2}\right)$ mixed strategy so that each town has $\frac{N}{2}$ population.
    - Assuming N is large so that the law of large numbers applies and that individual movement has negligible effect.
- Initial condition can be important.
  - Other applications: network effects

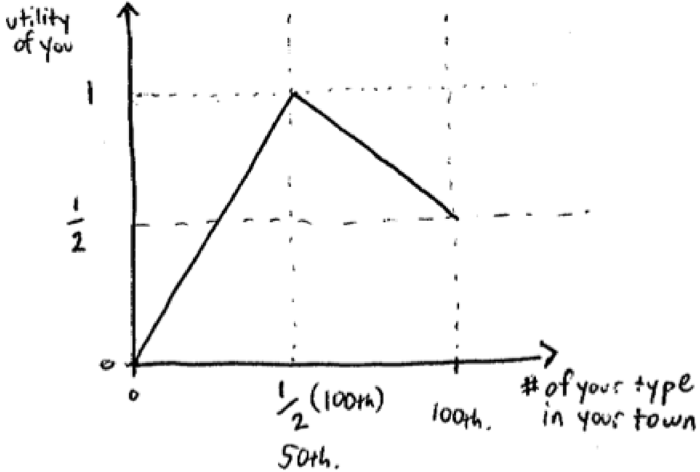
Choosing a Town (v2)

- A population of N can be divided equally into two types of people: tall and short.
- Each simultaneously chooses to live in one of two towns: East Town and West Town. Each town can hold $\frac{1}{2}N$ people.
  - If $n > \frac{1}{2}N$ chooses one town, then $\left(n - \frac{1}{2}N\right)$ people will be randomly chosen to be relocated to the other town.
- Let q_τ be the proportion of a town's residents who are of type τ. The payoff to an individual i of type τ who chooses to live in a town of q_τ is:

$$u_i^\tau\left(q_\tau\right) = 2q_\tau \mathcal{I}_{q_\tau \in \left[0, \frac{1}{2}\right]} + \left(\frac{3}{2} - q_\tau\right)\mathcal{I}_{q_\tau \in \left(\frac{1}{2}, 1\right]}$$

  - People perfer to live in mixed towns, but if they're going to live in a town that's not mixed, they'd rather live in a town in which they're the majority.

Choosing a Town (v2)



Choosing a Town (v2)

- NE:
    - (stable) All tall people in one town and short people in the other.
    - (weak) Each person plays a $\left(\frac{1}{2}, \frac{1}{2}\right)$ mixed strategy so that each town is perfectly mixed in realization.
        - Assuming N is large so that the law of large numbers applies and that individual movement has negligible effect.
- Even though everyone prefers mixed towns, the game results in segregation as stable outcome.
    - Conversely, observed segregation may not imply there is a preference for segregation.

Nash Equilibrium: Existence

Theorem

Every finite strategic form game possesses at least one NE.

Dynamic Games

Dynamic games are games where players are not moving simultaneously, but rather in a sequence over time.

Firm Entry

There are two firms, a potential entrant (E) and an industry incumbent (I).

- The potential entrant must decide whether to enter the market or not.
- If it stays out then it gains nothing and the incumbent firm gains 2.
- If the potential entrant enters, then
  - If the incumbent fights, both firms lose 1.
  - If the incumbent cooperates, both firms gain 1.

Firm Entry: Strategic Form

		Incumbent	
		F	C
Entrant	In	-1,-1	1,1
	Out	0,2	0,2

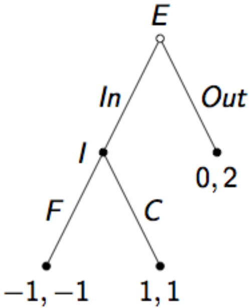
- PSNE: (In,C), (Out,F)

Firm Entry: Strategic Form

		Incumbent	
		F	C
Entrant	In	-1,-1	1,1
	Out	0,2	0,2

- PSNE: (In,C), (Out,F)
- Problem with (Out,F): the “threat” of Incumbent playing F upon entry is not credible.

Firm Entry: Extensive Form



Extensive Form Games with Perfect Information

Definition (Extensive Form Game with Perfect Information)

An extensive form game with perfect information is $\Gamma = \{\mathcal{N}, \mathcal{A}, \mathcal{X}, \mathcal{T}, P, A, u\}$, where

1. $\mathcal{N} = \{1, \ldots, N\}$ is a finite set of players,
2. $\mathcal{A}$ is the set of all possible actions,
3. $\mathcal{X}$ is the set of nodes, or histories,
    1. $\mathcal{X}$ contains $\emptyset$, the initial node, or empty history.
4. $\mathcal{T}$ is the set of terminal nodes,
5. $P : \mathcal{X} \backslash \mathcal{T} \rightarrow \mathcal{N}$ defines the player whose turn it is to move at x,
6. $A : \mathcal{X} \backslash \mathcal{T} \rightarrow \mathscr{P}(\mathcal{A})$ defines the set of available actions at x:

$$A(x) \equiv \{a \in \mathcal{A} : (x, a) \in \mathcal{X}\}$$

7. $u = (u_1, \ldots, u_N)$, where $u_i : \mathcal{T} \rightarrow \Re$ is the vNM utility function for player i.

Firm Entry: Extensive Form

$\Gamma = \{\mathcal{N}, \mathcal{A}, \mathcal{X}, \mathcal{T}, P, A, u\}$, where

- $\mathcal{N} = \{I, E\}$
- $\mathcal{A} = \{In, Out, F, C\}$
- $\mathcal{X} = \{\emptyset, In, Out, (In, F), (In, C)\}$
- $\mathcal{T} = \{Out, (In, F), (In, C)\}$
- $P(\emptyset) = E, P(In) = I$
- $A(\emptyset) = \{In, Out\}, A(In) = \{F, C\}$
- $u = (u_E, u_I)$, where

$$u_E(Out) = 0, u_E(In, F) = -1, u_E(In, C) = 1$$

$$u_I(Out) = 2, u_I(In, F) = -1, u_I(In, C) = 1$$

Extensive Form Game Strategies

Let $\mathcal{X}_i \equiv \{x \in \mathcal{X} : P(x) = i\}$ be the set of nodes at which it is player i's turn to move.

Definition (Pure Strategy in Extensive Form Games with Perfect Information)

Given a perfect information extensive form game Γ, a pure strategy for player i is $s_i : \mathcal{X}_i \to \mathcal{A}$ such that $s_i(x) \in A(x)\ \forall x \in \mathcal{X}_i$. Let S_i denote the set of all pure strategies for player i in Γ.

- A strategy is a **complete contingent plan** of what a player will do in *any* situation that could arise (i.e. at each decision node where it's her turn to move).

Strategic Form of Extensive Form Games

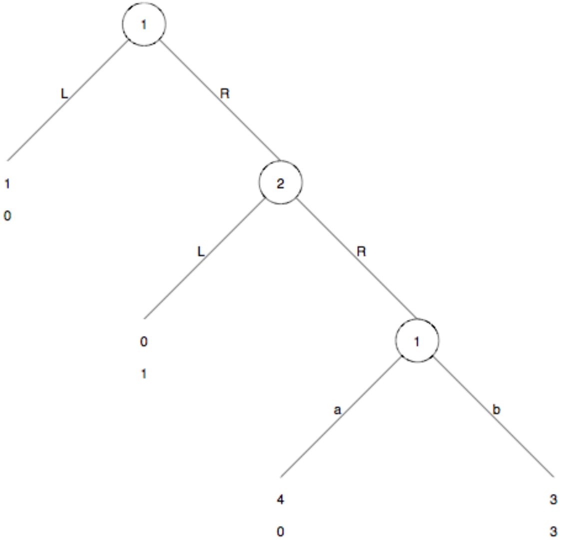
- Given the definition of extensive form strategies, the utility functions $u_i : \mathcal{T} \to \Re$ in $\Gamma = \{\mathcal{N}, \mathcal{A}, \mathcal{X}, \mathcal{T}, P, A, u\}$ can be written as payoff functions $u_i : S \to \Re$, where $S \equiv S_1 \times \cdots \times S_N$.
- The associated tuple $(S_i, u_i)_{i \in \mathcal{N}}$ is the strategic form of Γ.

Definition (Finite Game of Perfect Information)

Γ is called a finite game of perfect information if

- $\mathcal{X}$ is finite (which implies $\mathcal{A}$ is finite),
- or equivalently, S is finite in the strategic form of Γ.

An Extensive Form Game



An Extensive Form Game

Pure strategies:

- $S_1 = \left\{ s_1^1, s_1^2, s_1^3, s_1^4 \right\}$
    - $s_1^1 (\emptyset) = L, s_1^1 (R, R) = a$
    - $s_1^2 (\emptyset) = L, s_1^2 (R, R) = b$
    - $s_1^3 (\emptyset) = R, s_1^3 (R, R) = a$
    - $s_1^4 (\emptyset) = R, s_1^4 (R, R) = b$
- $S_2 = \left\{ s_2^1, s_2^2 \right\}$
    - $s_2^1 (R) = L$
    - $s_2^2 (R) = R$

Alternatively, we can write:

- $S_1 = \{(L, a), (L, b), (R, a), (R, b)\}$
- $S_2 = \{L, R\}$

An Extensive Form Game

Payoffs:

- $u_1\left(s_1^1, s_2^1\right) = 1, u_1\left(s_1^3, s_2^1\right) = 0, u_1\left(s_1^3, s_2^2\right) = 4, u_1\left(s_1^4, s_2^2\right) = 3, \ldots$
- $u_2\left(s_1^1, s_2^1\right) = 0, u_2\left(s_1^3, s_2^1\right) = 1, u_2\left(s_1^3, s_2^2\right) = 0, u_2\left(s_1^4, s_2^2\right) = 3, \ldots$

Strategic Form:

	L	R
La	1,0	1,0
Lb	1,0	1,0
Ra	0,1	4,0
Rb	0,1	3,3

An Extensive Form Game

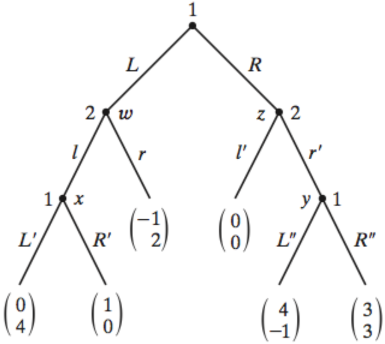
Payoffs:

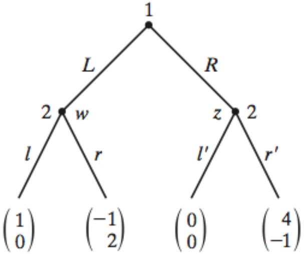
- $u_1\left(s_1^1, s_2^1\right) = 1, u_1\left(s_1^3, s_2^1\right) = 0, u_1\left(s_1^3, s_2^2\right) = 4, u_1\left(s_1^4, s_2^2\right) = 3, \ldots$
- $u_2\left(s_1^1, s_2^1\right) = 0, u_2\left(s_1^3, s_2^1\right) = 1, u_2\left(s_1^3, s_2^2\right) = 0, u_2\left(s_1^4, s_2^2\right) = 3, \ldots$

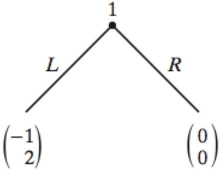
Strategic Form:

	L	R
La	1,0	1,0
Lb	1,0	1,0
Ra	0,1	4,0
Rb	0,1	3,3

Backward Induction (BI)







Lion and the Sheep

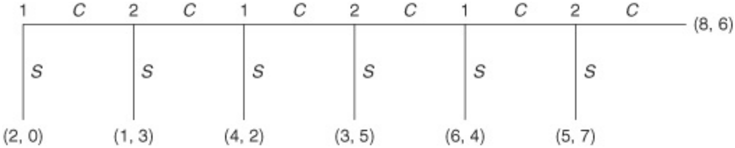
A pride of lions has captured a sheep. The lion pride has a rule: only the largest lion (head lion) can eat. The head lion, however, will fall into sleep after his meal, at which point the second largest lion can eat him and become the head lion. But if the second largest lion eats the largest lion, he too will fall in sleep, at which point the third largest lion can eat him, so on and so forth. Suppose the lions value life more than a good meal, will the sheep be eaten or not?

Nim

There are two piles of sticks. Two players alternate taking any number of sticks from any single one of the piles. The player that takes the last remaining stick wins.

Centipedes

Two players take turns choosing one of two actions each time, continue or stop. Player A starts with \$2 in her pile. Player B starts with \$0 in her pile. Each time player *i* says continue, \$1 is taken away from her pile, and \$2 are added to the other player's pile. The game automatically stops when both players have at least \$100 in their respective piles.



Corporate Spy

Two clothing companies, A and B, are simultaneously planning to enter a market. Each can decide whether to open a women's clothing store or a men's clothing store.

- If A opens a women's clothing store and B opens a men's clothing store, then A gets 150 in return and B gets 100. Vice versa.
- If both open a women's clothing store, then both get 75 in return.
- If both open a men's clothing store, then both gets 50 in return.

Now suppose B sends a spy to A to learn what A's plan is. A's CEO is informed that there is a spy in her company, but does not know who the spy is. What should A do?

Stackelberg Competition

Problem

Consider two firms engaging in Cournot competition subject to constant marginal cost c and market demand $P(Q) = \alpha - \beta Q$, where $Q = q_1 + q_2$.

- We know that $BR_i(q_j) = \max\left\{0, \frac{\alpha - c}{2\beta} - \frac{q_j}{2}\right\}$ and $q_1^* = q_2^* = \frac{\alpha - c}{3\beta}$.
- Now suppose firm 1 moves first. What is the resulting equilibrium?

Player 1's problem:

$$\begin{aligned} & \max_{q_1} q_1 \left(\alpha - \beta q_1 - \beta q_2^* - c\right) \\ = \quad & \max_{q_1} q_1 \left(\alpha - \beta q_1 - \beta \left(\frac{\alpha - c}{2\beta} - \frac{q_1}{2}\right) - c\right) \end{aligned}$$

$\Rightarrow$

$$(q_1^*, q_2^*) = \left(\frac{\alpha - c}{2\beta}, \frac{\alpha - c}{4\beta}\right)$$

Backward Induction Strategies

Let x be a penultimate node in Γ if all nodes immediately following x are end nodes. Let $s_{P(x)}(x)$ be an action that maximises player $P(x)$'s payoff from among $A(x)$. Let u_x denote the resulting payoff vector.

Definition (Backward Induction Strategy)

s is a BI strategy for the perfect information finite extensive form game Γ if it is derived as follows: remove the nodes and actions following each penultimate node x and assign the payoff u_x to x, which then becomes an end node in Γ. Repeat this process until an action has been assigned to every decision node.This yields a (BI) joint pure strategy s.

- This method of constructing a BI strategy is called the *backward induction algorithm*.

Nash Equilibrium in Finite Games of Perfect Information

Theorem

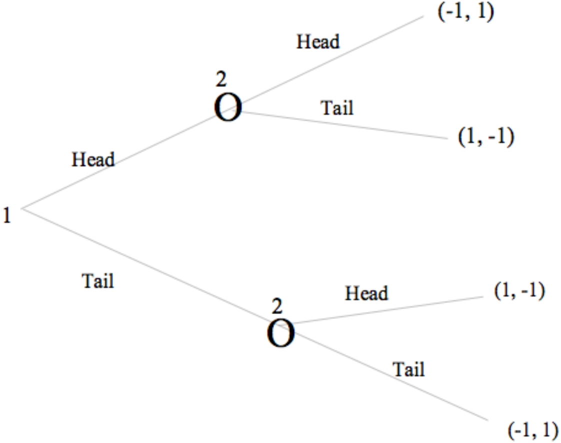
Given a finite game of perfect information Γ, s *is a BI strategy* $\Rightarrow$ s *is a NE.*

Theorem (Zermelo)

Every finite game of perfect information has a PSNE that can be derived through BI. Moreover, if no player has the same payoffs at any two terminal nodes, then BI results in a unique PSNE.

Matching Pennies with Perfect Information

Extensive Form

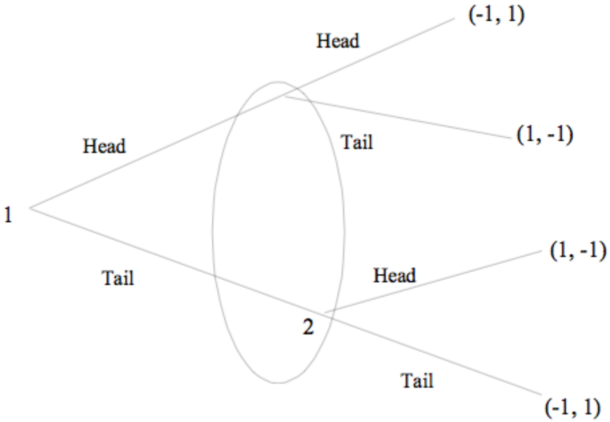


Strategic Form

1\2	HH	HT	TH	TT
H	-1,1	-1,1	1,-1	1,-1
T	1,-1	-1,1	1,-1	-1,1

Matching Pennies with Imperfect Information

Extensive Form



Strategic Form

1\2	H	T
H	-1,1	1,-1
T	1,-1	-1,1

Matching Pennies with Nature

In the matching pennies game, after both players have revealed their chosen sides of the coin, their payoffs are determined according to the following rule: another coin is flipped.

- If the Head comes up, the payoff matrix is:

1\2	H	T
H	-1,1	1,-1
T	1,-1	-1,1

- If the Tail comes up, the payoff matrix is:

1\2	H	T
H	1,-1	-1,1
T	-1,1	1,-1

Extensive Form Games

Definition (Extensive Form Game)

An extensive form game is $\Gamma = \{\mathcal{N}, \mathcal{A}, \mathcal{X}, \mathcal{T}, \mathcal{I}, P, A, u\}$, where $\mathcal{N}, \mathcal{A}, \mathcal{X}, \mathcal{T}, P, A, u$ are defined as in extensive form game with perfect information. In addition,

1. $\mathcal{X}$ contains an initial node x_0 at which Nature moves by randomly choosing an action from $A(x_0)$ according to probability distribution π,
2. $\mathcal{I}$ is a partition of $\mathcal{X} \backslash (\mathcal{T} \cup \{x_0\})$ that describes the information available to each player at each turn, and satisfies

$$\forall x' \in \mathcal{I}(x), \; P(x) = P(x') \text{ and } A(x) = A(x')$$

, where $\mathcal{I}(x)$ defnotes the member of $\mathcal{I}$ that contains x. $P(x)$ is assumed unable to distinguish between the nodes in $\mathcal{I}(x)$.

- Given $\mathcal{I}$, P and A can be defined as $P : \mathcal{I} \to \mathcal{N}$ and $A : \mathcal{I} \to \mathscr{P}(\mathcal{A})$.
- A game has perfect information if all information sets are singletons.

Extensive Form Game Strategies

Let $\mathcal{I}_i \equiv \{I \in \mathcal{I} : P(I) = i\}$ be the set of information sets nodes belonging to player i.

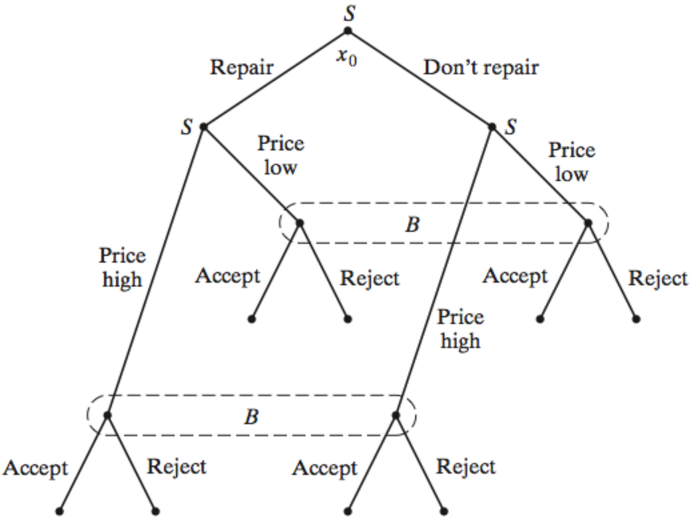
Definition (Pure Strategy in Extensive Form Games)

Given an extensive form game Γ, a pure strategy for player i is $s_i : \mathcal{I}_i \to \mathcal{A}$ such that $s_i(I) \in A(I)\ \forall I \in \mathcal{I}_i$.

Used Car

Consider the buyer and seller of a used car. The seller first chooses whether to repair the car and then chooses whether to price the car high or low. The buyer cannot observe whether the car has been repaired or not and will make a decision of whether to buy after being informed of its price.

Used Car


S
x_0
Repair
Don't repair
S
S
Price low
Price low
B
Price high
Accept
Reject
Accept
Reject
Price high
B
Accept
Reject
Accept
Reject

Used Car: Extensive Form

$\Gamma = \{\mathcal{N}, \mathcal{A}, \mathcal{X}, \mathcal{T}, \mathcal{I}, P, A, u\}$, where

- $\mathcal{N} = \{\mathrm{S,B}\}$
- $\mathcal{A} = \{\mathrm{R,DR,H,L,a,r}\}$[3]
- $\mathcal{X} = \{\emptyset,\mathrm{R,DR,(R,H),(R,L),(DR,H),(DR,L)}\}\cup\mathcal{T}$
- $\mathcal{T} = \{\mathrm{(R,H,a),(R,H,r),(R,L,a),(R,L,r),}$
  $\mathrm{(DR,H,a),(DR,H,r),(DR,L,a),(DR,L,r)}\}$
- $\mathcal{I} = \{\{\emptyset\},\{\mathrm{R}\},\{\mathrm{DR}\},\{\mathrm{(R,H),(DR,H)}\},\{\mathrm{(R,L),(DR,L)}\}\}$
  - $\mathcal{I}(\emptyset)=\{\emptyset\}$, $\mathcal{I}(\mathrm{R})=\{\mathrm{R}\}$, $\mathcal{I}(\mathrm{DR})=\{\mathrm{DR}\}$,
    $\mathcal{I}(\mathrm{(R,H)})=\{\mathrm{(R,H),(DR,H)}\}$, $\mathcal{I}(\mathrm{(DR,H)})=\{\mathrm{(R,H),(DR,H)}\}$,
    $\mathcal{I}(\mathrm{(R,L)})=\{\mathrm{(R,L),(DR,L)}\}$, $\mathcal{I}(\mathrm{(DR,L)})=\{\mathrm{(R,L),(DR,L)}\}$,
- P($\{\emptyset\}$)=S, P($\{\mathrm{R}\}$)=S, P($\{\mathrm{DR}\}$)=S,
  P($\{\mathrm{(R,H),(DR,H)}\}$)=B, P($\{\mathrm{(R,L),(DR,L)}\}$)=B
- A($\{\emptyset\}$)={R,DR}, A($\{\mathrm{R}\}$)={H,L}, A($\{\mathrm{DR}\}$)={H,L},
  A($\{\mathrm{(R,H),(DR,H)}\}$)={a,r}, A($\{\mathrm{(R,L),(DR,L)}\}$)={a,r}

---

[3] R : repair; DR : don't repair; H : price high; L : price low; a : Accept; r : reject.

Used Car: Pure Strategies

- $S_1 = \left\{s_1^1, s_1^2, s_1^3, s_1^4, s_1^5, s_1^6, s_1^7, s_1^8\right\}$
  - $s_1^1(\{\emptyset\}) = R, s_1^1(\{R\}) = H, s_1^1(\{DR\}) = H$
  - $s_1^2(\{\emptyset\}) = R, s_1^2(\{R\}) = H, s_1^2(\{DR\}) = L$
  - $s_1^3(\{\emptyset\}) = R, s_1^3(\{R\}) = L, s_1^3(\{DR\}) = H$
  - $s_1^4(\{\emptyset\}) = R, s_1^4(\{R\}) = L, s_1^4(\{DR\}) = L$
  - $s_1^5(\{\emptyset\}) = DR, s_1^5(\{R\}) = H, s_1^5(\{DR\}) = H$
  - $s_1^6(\{\emptyset\}) = DR, s_1^6(\{R\}) = H, s_1^6(\{DR\}) = L$
  - $s_1^7(\{\emptyset\}) = DR, s_1^7(\{R\}) = L, s_1^7(\{DR\}) = H$
  - $s_1^8(\{\emptyset\}) = DR, s_1^8(\{R\}) = L, s_1^8(\{DR\}) = L$
- $S_2 = \left\{s_2^1, s_2^2, s_2^3, s_2^4\right\}$
  - $s_2^1(\{(R,H),(DR,H)\}) = a, s_2^1(\{(R,L),(DR,L)\}) = a$
  - $s_2^2(\{(R,H),(DR,H)\}) = a, s_2^2(\{(R,L),(DR,L)\}) = r$
  - $s_2^3(\{(R,H),(DR,H)\}) = r, s_2^3(\{(R,L),(DR,L)\}) = a$
  - $s_2^4(\{(R,H),(DR,H)\}) = r, s_2^4(\{(R,L),(DR,L)\}) = r$

Extensive Form Game Strategies

Definition (Mixed Strategy in Extensive Form Games)

A mixed strategy for player i is $\sigma_i : S_i \rightarrow [0,1]$, such that $\sum_{s_i \in S_i} \sigma_i\left(s_i\right) = 1$.

- This is the same definition as mixed strategy in strategic form games

Definition (Behavioral Strategy in Extensive Form Games)

A behavioral strategy for player i is $\varsigma_i : \mathcal{A} \times \mathcal{I}_i \rightarrow [0,1]$, such that $\varsigma_i\left(a, I\right) = 0\ \forall a \notin A\left(I\right)$ and $\sum_{a \in A(I)} \varsigma_i\left(a, I\right) = 1$.

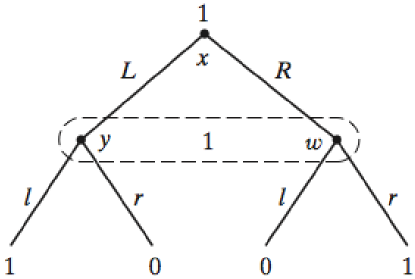
Perfect Recall

Definition (Perfect Recall)

An extensive form game has perfect recall if whenever two nodes x and $y = (x, a, a_1, \ldots, a_k)$ belong to a single player, then every node in the same information set as y is of the form $w = (z, a, a_1\prime, \ldots, a_l\prime)$ for some node z in the same information set as x.

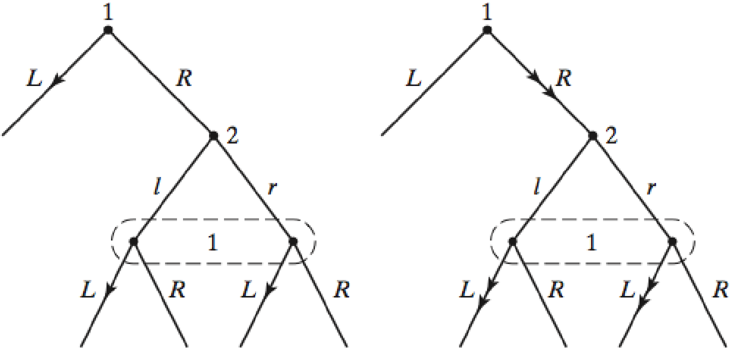
- A game has perfect recall if a player always remembers what she knew or did in the past. In particular, any two histories that belong to the same information set of a player can differ only in the actions taken by other players.
- Mixed strategies and behavioral strategies are **equivalent** in games with perfect recall.

Perfect Recall



Game without perfect recall

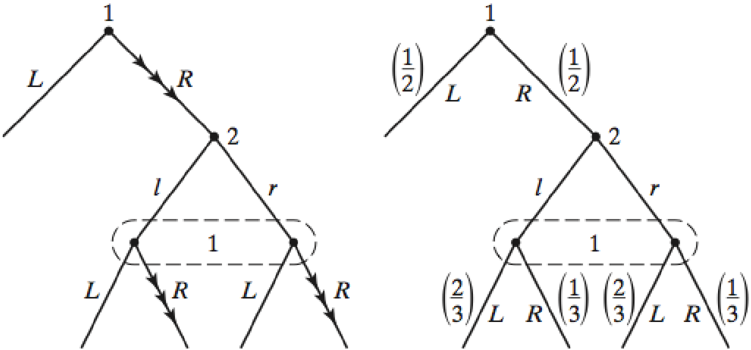
Mixed Strategy and Behavioral Strategy



(a) (b)

(a): $s_1^1 = (L, L)$; (b): $s_1^2 = (R, L)$

Mixed Strategy and Behavioral Strategy



(c) (d)

$$\text{(c): } s_1^3 = (R, R)$$

$$\text{(d): } \varsigma_1\left(\{\emptyset\}\right) = \tfrac{1}{2}L + \tfrac{1}{2}R,\ \varsigma_1\left(\{(R, l), (R, r)\}\right) = \tfrac{2}{3}L + \tfrac{1}{3}R$$

- ς_1 is equivalent to $\sigma_1 = \frac{1}{2}s_1^1 + \frac{1}{3}s_1^2 + \frac{1}{6}s_1^3$

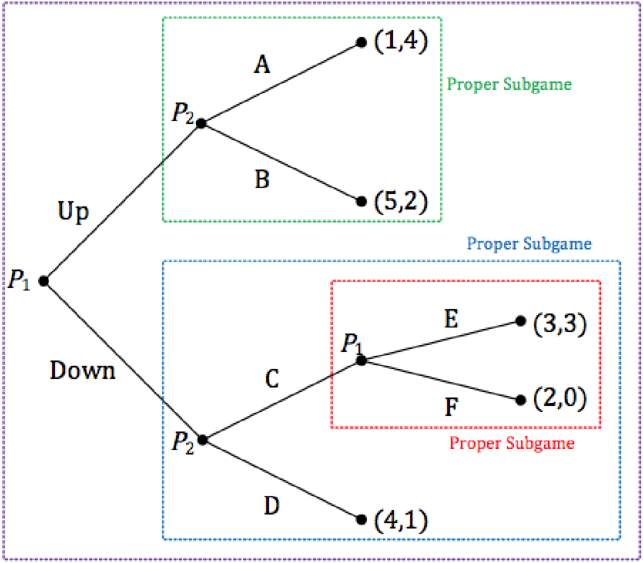
Subgames

Definition

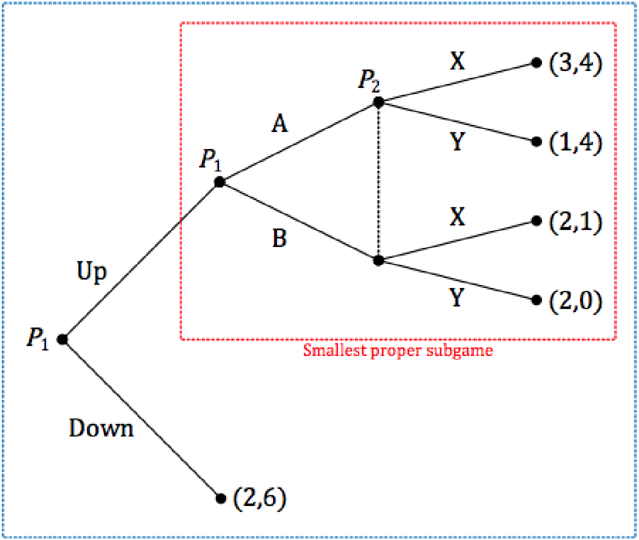
A node x is said to define a subgame of an extensive form game if $\mathcal{I}(x) = \{x\}$ and whenever y is a decision node following x, and z is in the information set containing y, then z also follows x.

- A subgame
  - starts from a single node
  - comprises all successors to that node
  - does not break up any information sets
- Any game is a subgame of itself. A subgame that is not the whole game itself is called a *proper* subgame.

Subgames

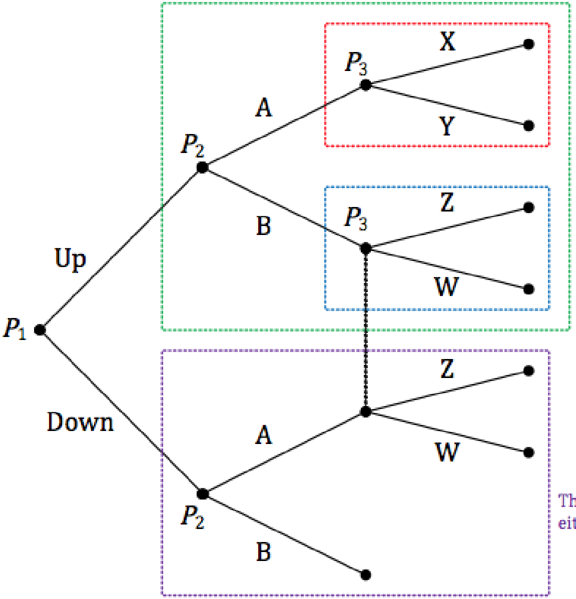

(1,4)
A
Proper Subgame
P_2
B
(5,2)
Up
Proper Subgame
P_1
E
(3,3)
P_1
Down
C
(2,0)
F
P_2
Proper Subgame
D
(4,1)

Subgames



The game as a whole is the second smallest subgame.

Subgames



Proper Subgame

This cannot be a proper subgame (We cannot break info. sets).

This cannot be a proper subgame either

This cannot be a proper subgame either

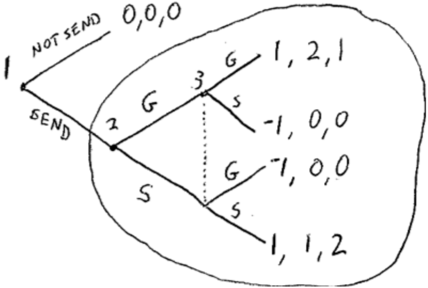
Subgame Perfect Equilibrium (SPE)

Definition

s is a subgame perfect equilibrium of Γ if s induces a NE in every subgame of Γ.

Matchmaking

Lucy is considering whether to send her friends, Tom and Aimee, onto a dinner date. If she does, Tom and Aimee need to simultaneously decide whether they would like to go for Sushi or Gyro.

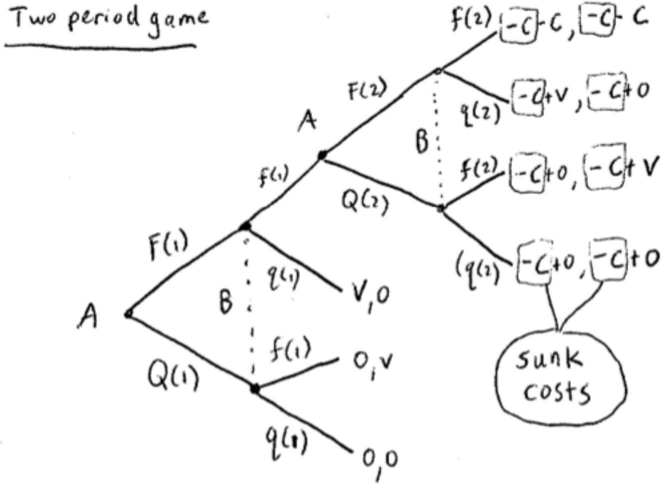


- SPE: (Send,S,S), (Send,G,G), (Not Send,$\left(\frac{2}{3},\frac{1}{3}\right)$,$\left(\frac{1}{3},\frac{2}{3}\right)$)

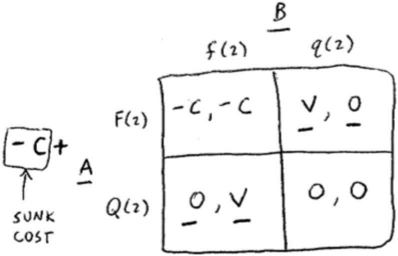
War of Attrition

Two players choose whether to Fight or Quit in each period. The game ends as soon as one player chooses to Quit. If one player Quits and the other player chooses to Fight, then the other player gets v. If both Fight, then both pay a cost of c ($v > c$). If both Quit at once, then both get 0.

War of Attrition: Two Period



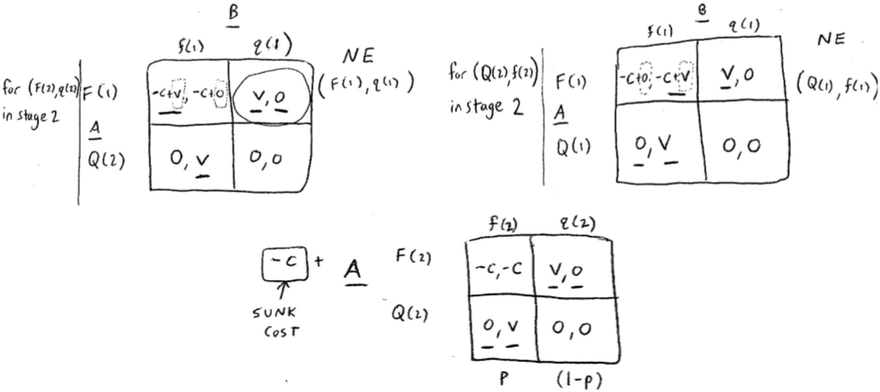
War of Attrition: Two Period



	f(2)	q(2)
F(2)	-C, -C	$\underline{V}$, $\underline{0}$
Q(2)	$\underline{0}$, $\underline{V}$	0, 0

- NE: (F(2),q(2)), (Q(2),f(2)), (p^*,p^*), where $p^* = \frac{v}{v+c}$

War of Attrition: Two Period



for (F(2),q(2)) in stage 2

A \ B	f(1)	q(1)
F(1)	-c+v, -c+0	v, 0
Q(2)	0, v	0, 0

NE (F(1), q(1))

for (Q(2),f(2)) in stage 2

A \ B	f(1)	q(1)
F(1)	-c+0, -c+v	v, 0
Q(1)	0, v	0, 0

NE (Q(1), f(1))

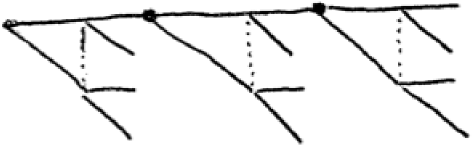
$\boxed{-c}$ + A

SUNK Cost

A	f(2)	q(2)
F(2)	-c, -c	v, 0
Q(2)	0, v	0, 0
	p	(1-p)

- SPE: ((F(1),F(2)),(q(1),q(2))), ((Q(1),Q(2)),(f(1),f(2))), ((p^*,p^*),(p^*,p^*))

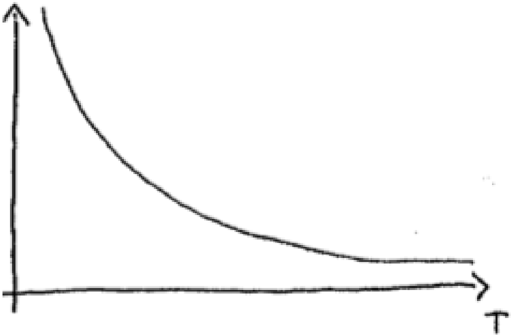
War of Attrition: Infinite Period



- SPE: ((F(1),F(2),. . .),(q(1),q(2),. . .)), ((Q(1),Q(2),. . .),(f(1),f(2),. . .)), ((p^*,p^*,. . .),(p^*,p^*,. . .))

War of Attrition: Infinite Period

- Prob(length of the game) if the mixed strategy SPE is played:



Subgame Perfect Equilibrium: Existence

Theorem

For every finite game of perfect information, the set of BI strategies coincide with the set of pure strategy SPEs.

Theorem

Every finite game has a SPE.

Games of Incomplete Information

Definition (Game of Complete Information)

A complete information game is one where all players' payoff functions are common knowledge

In games of **incomplete information**,

- Players can have access to private information not observed by other players. This is modeled as each player having a privately observed type $\theta_i \in \Theta_i$.
- A strategy describes what a player would do given her type: $\sigma_i(\theta_i)$
- Individual payoff depends both on the chosen strategies and the types of all players: $u_i : \Delta \times \Theta \to \Re$, where $\Theta = \Theta_1 \times \cdots \times \Theta_N$.

Bayesian Nash Equilibrium (BNE)

Definition

A strategy profile is a Bayesian Nash Equilibrium of a game of incomplete information if for each player i and type θ_i,

$$\begin{aligned} & \mathbb{E}_{\theta_{-i}}\left[u_i\left(\sigma_i^*\left(\theta_i\right), \sigma_{-i}^*\left(\theta_{-i}\right), \theta_i, \theta_{-i}\right)\right] \\ \geq \quad & \mathbb{E}_{\theta_{-i}}\left[u_i\left(\sigma_i\left(\theta_i\right), \sigma_{-i}^*\left(\theta_{-i}\right), \theta_i, \theta_{-i}\right)\right] \; \forall \sigma_i\left(\theta_i\right) \in \Delta_i \end{aligned}$$

- Let $p_i\left(\theta_{-i} \middle| \theta_i\right)$ be player i's belief about the distribution of the other players' types. Then

$$\begin{aligned} & \mathbb{E}_{\theta_{-i}}\left[u_i\left(\sigma_i^*\left(\theta_i\right), \sigma_{-i}^*\left(\theta_{-i}\right), \theta_i, \theta_{-i}\right)\right] \\ = \quad & \sum_{\theta_{-i} \in \Theta_{-i}} p_i\left(\theta_{-i} \middle| \theta_i\right) u_i\left(\sigma_i^*\left(\theta_i\right), \sigma_{-i}^*\left(\theta_{-i}\right), \theta_i, \theta_{-i}\right) \end{aligned}$$

- When there exists an objective empirical distribution over players' types, $p_i\left(\theta_{-i} \middle| \theta_i\right) = p\left(\theta_{-i}\right)$.

Gift Game

Player 1 is considering whether to send player 2 a gift. However, player 2 does not know whether player 1 is a friend or an enemy. If player 1 is an enemy, her gift may not be good (e.g. frog in a box)

- If player 1 is a friend:

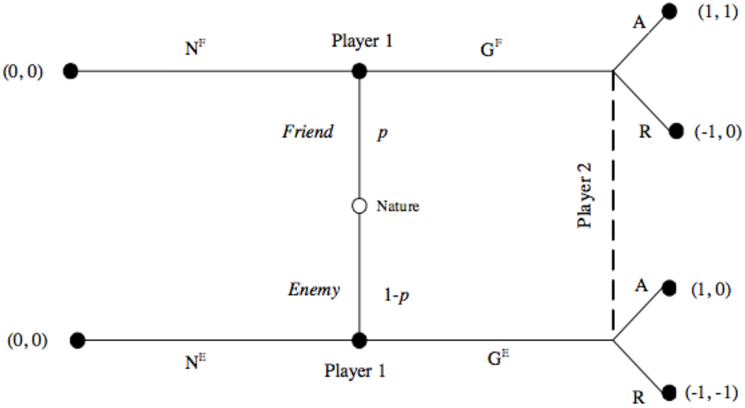
1\2	A	R
G^F	$1, 1$	$-1, 0$
N^F	$0, 0$	$0, 0$

- If player 1 is an enemy:

1\2	A	R
G^E	$1, 0$	$-1, -1$
N^E	$0, 0$	$0, 0$

- Player 2 believes player 1 is a friend with probability p.

Gift Game: Extensive Form



Gift Game: Strategic Form

$1\backslash 2$	A	R
$G^F G^E$	$1, p$	$-1, -(1-p)$
$G^F N^E$	p, p	$-p, 0$
$N^F G^E$	$1-p, 0$	$-(1-p), -(1-p)$
$N^F N^E$	$0, 0$	$0, 0$

- BNE: $\left(G^F G^E, A\right), \left(N^F N^E, R\right)$

An Incomplete Information Game

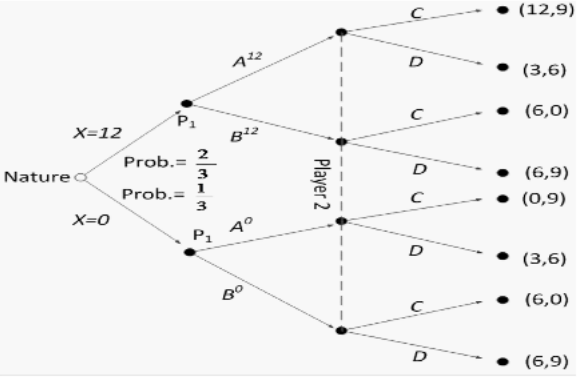
Player 1 and player 2 simultaneously choose an action (respectively from $\{A, B\}$ and $\{C, D\}$), while player 1 observes the value of x and player 2 only knows x's probability distribution.

1\2	C	D
A	x,9	3,6
B	6,0	6,9

$$\text{where } x = \begin{cases} 12 & \text{with prob. } \frac{2}{3} \\ 0 & \text{with prob. } \frac{1}{3} \end{cases}$$

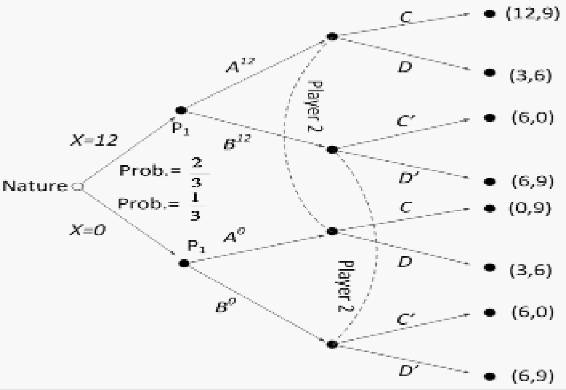
An Incomplete Information Game

Player 2 does not observe player 1's type or action:



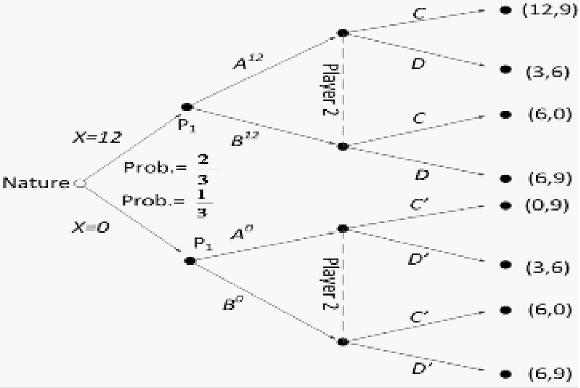
An Incomplete Information Game

If player 2 observes player 1's action but not her type:



An Incomplete Information Game

If player 2 observes player 1's type but not her action:



An Incomplete Information Game

1\2	C	D
$A^{12}A^0$	8,9	3,6
$A^{12}B^0$	10,6	4,7
$B^{12}A^0$	4,3	5,8
$B^{12}B^0$	6,0	6,9

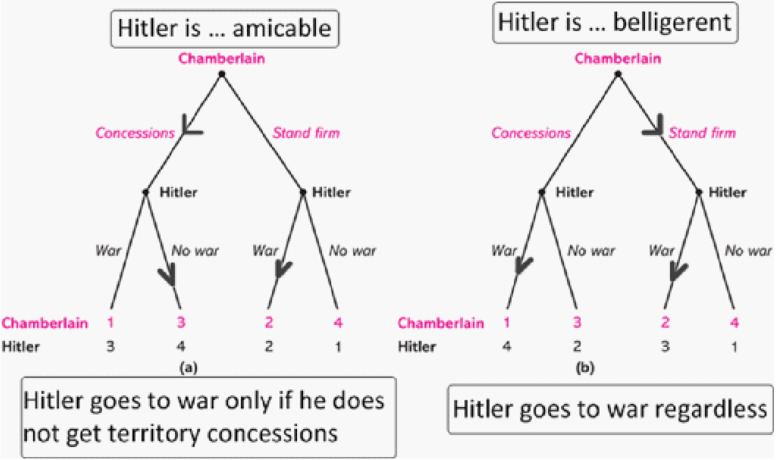
- BNE: $\left(B^{12}B^0, D\right)$

The Munich Agreement

In 1938, Hitler has invaded Czechoslovakia, and UK's prime minister, Chamberain, must decide whether to concede on such annexation to Germany or stand firm not allowing the occupation.

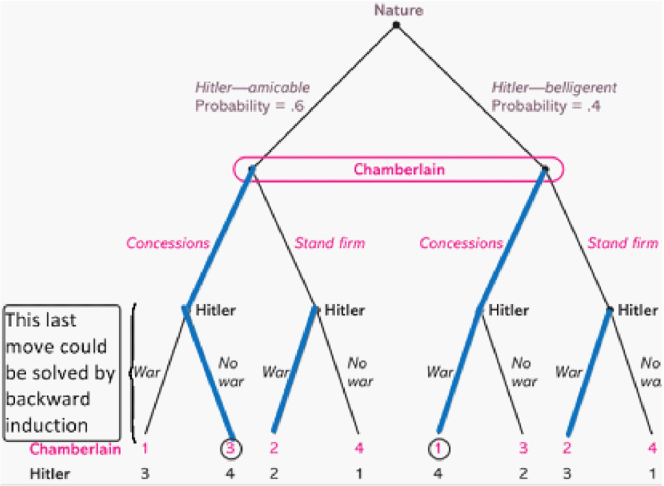
The Munich Agreement

Chamberlain doesn't know Hitler's exact incentives, but knows that Hitler can either be belligerent or amicable.



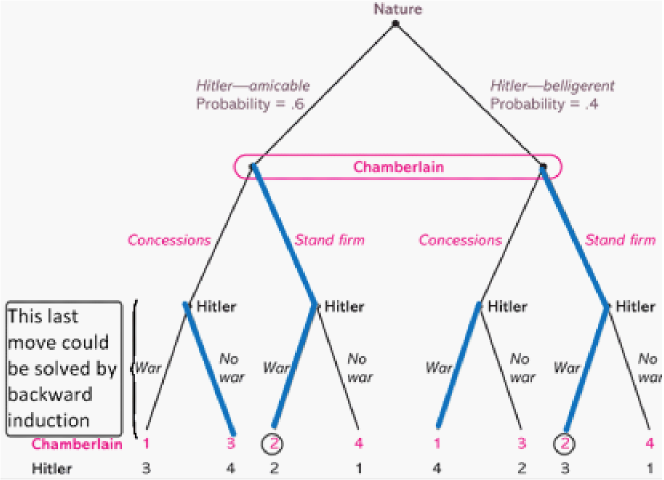
The Munich Agreement

- If Chamberlain chooses to give concessions:



The Munich Agreement

- If Chamberlain stands firm:



The Munich Agreement

BNE:

- Chamberlain: concessions
- Hitler:
    - When amicable, NW after concessions, W after standing firm
    - When belligerent, W after concessions, W after standing firm

Cournot with Incomplete Information about Firm Costs

- Consider Cournot competition in which firm 1's marginal cost $MC_1 = 0$ and is common knowledge. Firm 2's marginal cost MC_2 can be either high ($\frac{1}{4}$) or low (0) and is private knowledge.
- Firm 1 believes MC_2 has the following distribution:

$$MC_2 = \begin{cases} 0 & \text{with prob. } \frac{1}{2} \\ \frac{1}{4} & \text{with prob. } \frac{1}{2} \end{cases}$$

  This belief is common knowledge.
- Market demand:

$$P = 1 - Q$$

Cournot with Incomplete Information about Firm Costs

- Remember in Cournot games

$$BR_i(q_j) = \max\left\{0, \frac{\alpha - c}{2\beta} - \frac{q_j}{2}\right\}$$

- In this case, firm 2's best response:

$$BR_2(q_1) = \begin{cases} \frac{1}{2} - \frac{1}{2}q_1 & MC_2 = 0 \\ \frac{3}{8} - \frac{1}{2}q_1 & MC_2 = \frac{1}{4} \end{cases}$$

Cournot with Incomplete Information about Firm Costs

- Firm 1:

$$u_1\left(q_1, q_2^L, q_2^H\right) = \frac{1}{2}q_1\left(1 - q_1 - q_2^L\right) + \frac{1}{2}q_1\left(1 - q_1 - q_2^H\right)$$

$\Rightarrow$

$$\begin{aligned} BR_1\left(q_2^L, q_2^H\right) &= \frac{1}{2} - \frac{1}{2}q_2^L - \frac{1}{2}q_2^H \\ &= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2}q_1\right) - \frac{1}{2}\left(\frac{3}{8} - \frac{1}{2}q_1\right) \end{aligned}$$

- BNE: $\left(q_1, q_2^L, q_2^H\right) = \left(\frac{3}{8}, \frac{5}{16}, \frac{3}{16}\right)$

Cournot with Incomplete Information about Market Demand

- Consider Cournot competition in which market demand can be either high ($p(Q) = 10 - Q$) or low ($p(Q) = 5 - Q$).
- Firm 1 knows the actual demand curve. Firm 2 does not, but believes the demand curve has the following probability distribution:

$$p(Q) = \begin{cases} 10 - Q & \text{with prob. } \frac{1}{2} \\ 5 - Q & \text{with prob. } \frac{1}{2} \end{cases}$$

  This belief is common knowledge.
- Both firms have marginal cost $= 1$.

Cournot with Incomplete Information about Market Demand

- Firm 1's best response:

$$BR_1\left(q_2\right)=\begin{cases}\frac{9}{2}-\frac{1}{2}q_2 & \text{High Demand}\\ 2-\frac{1}{2}q_2 & \text{Low Demand}\end{cases}$$

- Firm 2:

$$u_2\left(q_1^L,q_2^H,q_2\right)=\frac{1}{2}q_1\left(10-q_1^H-q_2-1\right)+\frac{1}{2}q_1\left(5-q_1^L-q_2-1\right)$$

$\Rightarrow$

$$BR_2\left(q_1^L,q_1^H\right)\;=\;3.25-0.25\left(q_1^L+q_1^H\right)$$

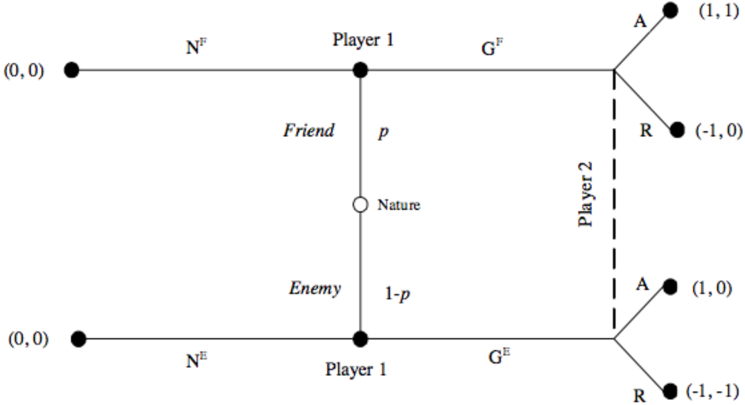
- BNE: $\left(q_1^L,q_1^H,q_2\right)=(3.416,0.916,2.167)$

Bayesian-Nash Equilibrium: Existence

Theorem

Every finite game of incomplete information possesses at least one BNE.

Gift Game



- BNE: $\left(G^F G^E, A\right)$, $\left(N^F N^E, R\right)$

Sequential Rationality

- At every information set at which every player is called to move, every player chooses the action that maximizes her expected payoff, given that all other players will do the same, and given her own beliefs about the other players' types.
- Beliefs are derived from Bayes' rule when possible

$$P\left(A|B\right)=\frac{P\left(B|A\right)P\left(A\right)}{P\left(B\right)}$$

Sequential Rationality

- Beliefs are derived from Bayes' rule when possible
    - Example: in the gift game, let α^F, α^E be, respectively, the prob. that player 1 plays G^F when Friend and α^E when Enemy. Let μ be the player 2's belief that player 1 is a Friend conditonal on receiving a gift. Then

$$\mu = \frac{p\alpha^F}{p\alpha^F + (1-p)\,\alpha^E}$$

    - If player 2's information set is not reached $(p\alpha^F + (1-p)\,\alpha^E = 0)$, then μ is called an **off-of-equilibrium** belief. An arbitrary value of $\mu \in [0,1]$ can be assigned.

Perfect Bayesian Equilibrium (PBE)

Definition

A strategy profile σ and beliefs μ are a Perfect Bayesian Equilibrium if

1. Each player's strategy specifies optimal actions at her each information set, given the strategies of the other players and her beliefs.
2. The beliefs are consistent with Bayes' rule, whenever possible.

- **Separating PBE**: different types of the privately informed player behave differently.
- **Pooling PBE**: all types of the privately informed player behave similarly.

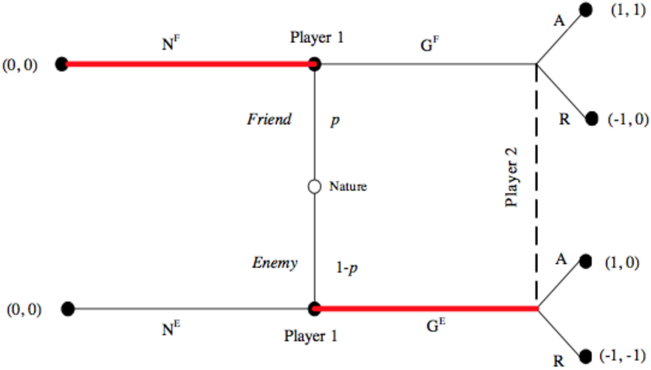
Finding PBE

1. Specify a strategy for the privately informed player, either separating or pooling.
2. Update the uninformed player's beliefs using Bayes' rule, when possible.
3. Given the uninformed playerís updated beliefs, find his optimal response.
4. Given the optimal response of the uninformed player, find the optimal actions for the informed player.
5. Check if the resulting strategy for the informed player coincides with the strategy suggested in step 1.
    - If this is the case, we say that *this strategy can be supported as part of a PBE of the game.*
    - Otherwise, we say that *this strategy cannot be sustained as part of a PBE.*

Gift Game

Exercise

Separating equilibrium with $N^F G^E$



Gift Game

Exercise

Separating equilibrium with $N^F G^E$

- Player 2's belief conditional on receiving a gift:

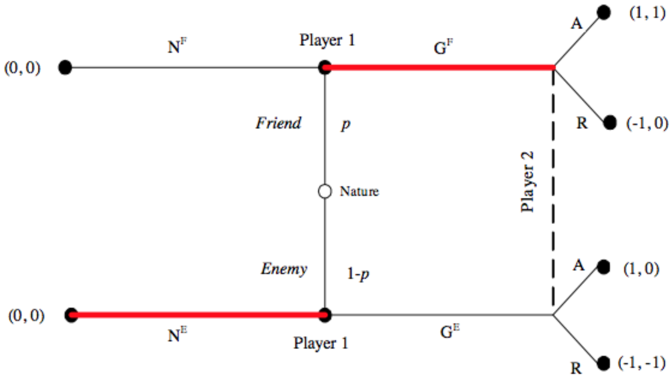
$$\mu = \frac{p\alpha^F}{p\alpha^F + (1-p)\,\alpha^E} = \frac{p \times 0}{p \times 0 + (1-p) \times 1} = 0$$

- Player 2's optimal action: A
- Given player 2's optimal action, player 1's optimal action is G^F when she is a friend
- The strategy $N^F G^E$ **can not** be sustained as part of a PBE

Gift Game

Exercise

Separating equilibrium with $G^F N^E$



Gift Game

Exercise

Separating equilibrium with $G^F N^E$

- Player 2's belief conditional on receiving a gift:

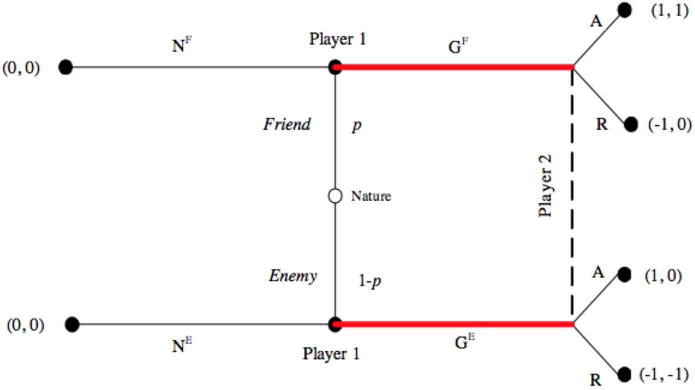
$$\mu = \frac{p\alpha^F}{p\alpha^F + (1-p)\,\alpha^E} = 1$$

- Player 2's optimal action: A
- Given player 2's optimal action, player 1's optimal action is G^E when she is an enemy
- The strategy $G^F N^E$ **can not** be sustained as part of a PBE

Gift Game

Exercise

Pooling equilibrium with $G^F G^E$



Gift Game

Exercise

Pooling equilibrium with $G^F G^E$

- Player 2's belief conditional on receiving a gift:

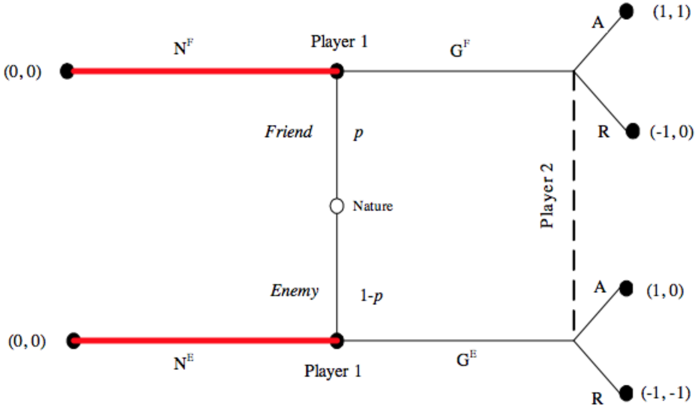
$$\mu = \frac{p\alpha^F}{p\alpha^F + (1-p)\,\alpha^E} = p$$

- Player 2's optimal action: A
- Given player 2's optimal action, player 1's optimal action is G^F when she is a friend and G^E when she is an enemy
- The strategy profile $\left(G^F G^E, A\right)$ **can** be supported as a PBE

Gift Game

Exercise

Pooling equilibrium with $N^F N^E$



Gift Game

Exercise

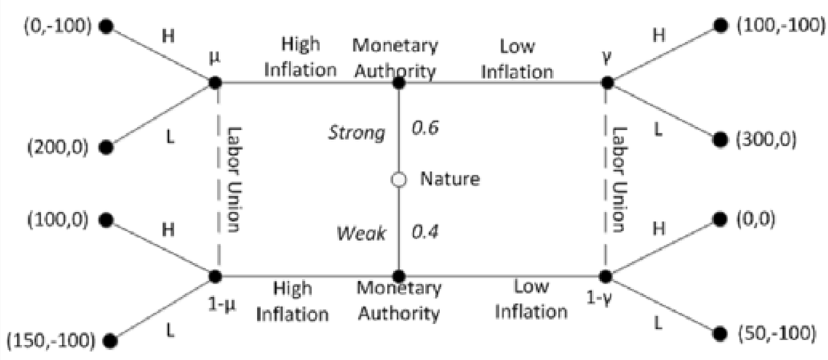
Pooling equilibrium with $N^F N^E$

- μ is an off-of-equilibrium belief and is undetermined.
- Player 2's optimal action: A
- Given player 2's optimal action, player 1's optimal action is G^F when she is a friend and G^E when she is an enemy
- The strategy $N^F N^E$ **can not** be sustained as part of a PBE

Monetary Authority

- A monetary authority, such as a central bank, can be either strong or weak.
- After knowing its type, the monetary authority makes an annoucement that the inflation expectation is either high or low.
- A labor union, observing the message sent by the monetary authority, decides whether to ask for high or low salary raises.

Monetary Authority



Monetary Authority

Exercise

Separating equilibrium with (Low Inflation, High Inflation)

- Player 2's beliefs

$$\mu = \frac{p\alpha^S}{p\alpha^S + (1-p)\,\alpha^W} = 0, \gamma = \frac{p\left(1-\alpha^S\right)}{p\left(1-\alpha^S\right) + (1-p)\left(1-\alpha^W\right)} = 1$$

- Player 2's optimal action: H after high inflation, L after low inflation
- Player 1's optimal action: LI^S when strong and HI^W when weak
- The strategy profile $\left(LI^S HI^W, H^{HI} L^{LI}\right)$ **can** be supported as a PBE

Monetary Authority

Exercise

Separating equilibrium with (High Inflation, Low Inflation)

- Player 2's beliefs: $\mu = 1, \gamma = 0$
- Player 2's optimal action: L after high inflation, H after low inflation
- Player 1's optimal action: HI^S when strong and HI^W when weak
- The strategy $HI^S LI^W$ **can not** be sustained as part of a PBE

Monetary Authority

Exercise

Pooling equilibrium with (High Inflation, High Inflation)

- Player 2's beliefs: $\mu = 0.6, \gamma \in [0, 1]$ (γ is an off-of-equilibrium belief and hence undetermined)
- If $\gamma < \frac{1}{2}$
  - Player 2's optimal action: L after high inflation, H after low inflation
  - Player 1's optimal action: HI^S when strong and HI^W when weak
- If $\gamma \geq \frac{1}{2}$
  - Player 2's optimal action: L after high inflation, L after low inflation
  - Player 1's optimal action: LI^S when strong and HI^W when weak
- The strategy $HI^S HI^W$ **can** be supported as part of a PBE **when** $\gamma < \frac{1}{2}$

Monetary Authority

Exercise

Pooling equilibrium with (Low Inflation, Low Inflation)

- Player 2's beliefs: $\mu \in [0,1], \gamma = 0.6$
- If $\mu < \frac{1}{2}$
  - Player 2's optimal action: H after high inflation, L after low inflation
  - Player 1's optimal action: LI^S when strong and HI^W when weak
- If $\mu \geq \frac{1}{2}$
  - Player 2's optimal action: L after high inflation, L after low inflation
  - Player 1's optimal action: LI^S when strong and HI^W when weak
- The strategy $LI^S LI^W$ **can not** be sustained as part of a PBE

Reference

The lecture slides draw from the following sources

- Jehle, G. A. and P. J. Reny. 2011. “Advanced Microeconomic Theory,” Prentice Hall, 3e.
- Graham, B. 2013. “Advanced Microeconomics II,” lecture notes at the Wang Yanan Institute for Studies in Economics, Xiamen University
- Polak, B. 2007. “Game Theory,” recorded lectures: http://oyc.yale.edu/economics/econ-159
- Mobius, M. M. 2008. “Advanced Game Theory,” lecture notes: http://isites.harvard.edu/icb/icb.do?keyword=k40228
- Weatherson, B. 2011. “Lecture Notes on Game Theory,” http://brian.weatherson.org
- Kartik, N. 2009. “Lecture Notes for 1st Year Ph.D. Game Theory,” http://econweb.ucsd.edu/∼jsobel/200Cs09/200Cs09home.htm